Fourier series

#1 (a.) $f(t)$ - 2π-periodic; $f(t) = t\sin 2t$ for $0 < t < 2\pi$.

Fourier coefficients:

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} t\sin 2t\,dt = \frac{-1}{4\pi}\int_0^{2\pi} t\,d\cos 2t =$$

$$= -\frac{t}{4\pi}\cos 2t\Big|_{t=0}^{2\pi} + \underbrace{\frac{1}{4\pi}\int_0^{2\pi}\cos 2t\,dt}_{=0} =$$

$$= -\frac{1}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} t\sin 2t\cos nt\,dt$$

$$\sin 2t\cos nt = \frac{1}{2}\sin(n+2)t + \frac{1}{2}\sin(n-2)t \Rightarrow$$

$$\Rightarrow a_n = \frac{1}{2\pi}\int_0^{2\pi} t\sin(n+2)t\,dt + \frac{1}{2\pi}\int_0^{2\pi} t\sin(n-2)t\,dt.$$

For integer $k \neq 0$:

$$\frac{1}{2\pi}\int_0^{2\pi} t\sin kt\,dt = \frac{-1}{2\pi k}\int_0^{2\pi} t\,d\cos kt =$$

$$= -\frac{t}{2\pi k}\cos kt\Big|_{t=0}^{2\pi} + \underbrace{\frac{1}{2\pi k}\int_0^{2\pi}\cos kt\,dt}_{=0} =$$

$$= -\frac{1}{k}$$

Therefore

$$n \neq 2 \Rightarrow a_n = \frac{-1}{n+2} + \frac{-1}{n-2} = \frac{-2n}{n^2-4}$$

$$n = 2 \Rightarrow a_2 = -\frac{1}{4}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} t \sin 2t \sin nt \, dt$$

$$\sin 2t \sin nt = \frac{1}{2}\cos(n-2)t - \frac{1}{2}\cos(n+2)t \Rightarrow$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} t\cos(n-2)t \, dt - \frac{1}{2\pi}\int_0^{2\pi} t\cos(n+2)t \, dt$$

For integer $k \neq 0$:

$$\frac{1}{2\pi}\int_0^{2\pi} t\cos kt \, dt = \frac{1}{2k\pi}\int_0^{2\pi} t \, d\sin kt =$$

$$= \underbrace{\frac{t}{2k\pi}\sin kt \Big|_0^{2\pi}}_{=0} - \underbrace{\frac{1}{2k\pi}\int_0^{2\pi}\sin kt \, dt}_{=0} = 0$$

$$\Rightarrow b_n = 0 \qquad n \neq 2$$

$$b_2 = \frac{1}{2\pi}\int_0^{2\pi} t\,dt = \frac{1}{2\pi}\,\frac{1}{2}t^2\Big|_0^{2\pi} = \pi$$

Finally

$$\left\{ f(t) \sim -\frac{1}{2} - \frac{1}{4}\cos 2t + \pi \sin 2t + \sum_{\substack{n>0 \\ n\neq 2}} \frac{2n}{4-n^2}\cos nt \right.$$

(b.) $f(t)$ odd and 2π-periodic,
$f(t) = t\sin 2t \,,\quad 0 \le t < \pi$.

It should be just sin-Fourier series:

$$b_n = \frac{2}{\pi}\int_0^{\pi} t \sin 2t \sin nt\,dt =$$

$$= \frac{1}{\pi}\int_0^{\pi} t\cos(n-2)t\,dt - \frac{1}{2\pi}\int_0^{\pi} t\cos(n+2)t\,dt$$

For $k \neq 0$:

$$\frac{1}{\pi}\int_0^{\pi} t\cos kt\,dt = \underbrace{\frac{t}{k\pi}\sin kt\Big|_0^{\pi}}_{=0} - \frac{1}{k\pi}\int_0^{\pi}\sin kt\,dt$$

$$= \frac{1}{k^2\pi}(1-\cos k\pi) = \begin{cases} 0 & k\text{-even} \\ \dfrac{2}{k^2\pi} & k\text{-odd} \end{cases}$$

$\Rightarrow$ For $n \neq 2$:

$b_n = 0$ if n is even ;

$$b_n = \frac{2}{\pi}\left(\frac{1}{(n-2)^2} - \frac{1}{(n+2)^2}\right) \quad \text{if } n \text{ is odd}$$

$$b_2 = \frac{1}{\pi}\int_0^\pi t\,dt = \frac{\pi}{2}$$

Finally

$$\begin{cases} f(t) \sim \frac{2}{\pi}\left(1 - \frac{1}{9}\right)\sin t + \frac{\pi}{2}\sin 2t + \\ + \frac{2}{\pi}\sum_{l=1}^{\infty}\left[\frac{1}{(2l-1)^2} - \frac{1}{(2l+3)^2}\right]\sin(2l+1)t \end{cases}$$

(c) $f(t)$ - even and 2π-periodic

$f(t) = t \sin 2t \quad , \quad 0 \leq t < \pi$

$$a_0 = \frac{1}{\pi}\int_0^\pi t \sin 2t\,dt = -\frac{1}{2}$$

$$a_n = \frac{2}{\pi}\int_0^\pi t \sin 2t \cos nt\,dt =$$

$$= \frac{1}{\pi}\int_0^\pi t \sin(n+2)t\,dt + \frac{1}{\pi}\int_0^\pi t \sin(n-2)t\,dt$$

$$n \neq 2 \Rightarrow a_n = (-1)^{n+1} \left(\frac{1}{n+2} + \frac{1}{n-2} \right) =$$

$$= (-1)^{n+1} \frac{2n}{n^2 - 4}$$

$$n = 2 \Rightarrow a_2 = -\frac{1}{4}$$

Finally:

$$f(t) \sim -\frac{2}{3} \cos t - \frac{1}{4} \cos 2t +$$

$$+ \sum_{n=3}^{\infty} (-1)^{n+1} \frac{2n}{n^2-4} \cos nt$$

#2. Here you do NOT need use the formulas for the Fourier coefficients.

a. $f(t) = \cos^3 t =$

$$= \underbrace{\frac{1}{2}(1+\cos 2t)}_{\cos^2 t} \cdot \cos t =$$

$$= \frac{1}{2}\cos t + \frac{1}{2}\cos 2t \cos t =$$

$$= \frac{1}{2}\cos t + \frac{1}{4}\cos t + \frac{1}{4}\cos 3t =$$

$$= \frac{3}{4}\cos t + \frac{1}{4}\cos 3t$$

b. $f(t) = \cos 3t$ – this is already Fourier series with just one nonzero entry.

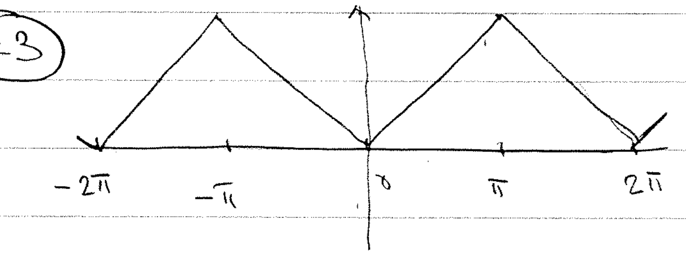
c. $f(t) = (\cos 2t + \sin 2t)^2 =$

$$= (\cos 2t)^2 + 2\cos 2t \sin 2t + (\sin 2t)^2 =$$

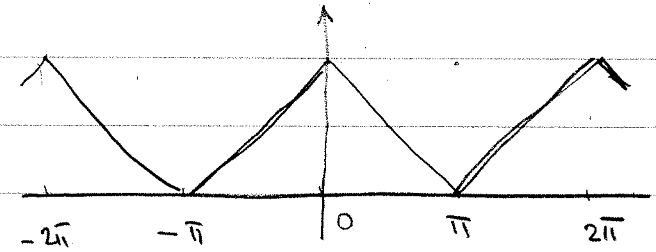
$$= 1 + 2\sin 4t$$

Fourier series

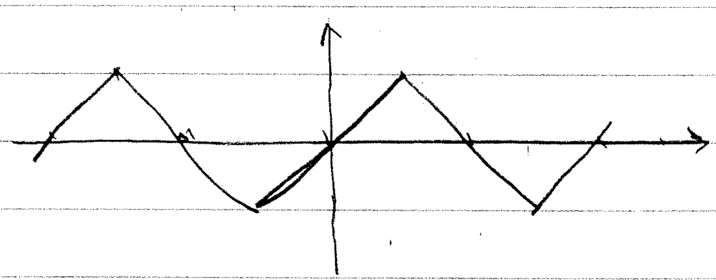
#3



$f_1(t)$



$f_2(t)$



$f_3(t)$

$$f_2(t) = f_1(t+\pi)$$

$$f_3(t) = -\frac{\pi}{2} + f_1\left(t+\frac{\pi}{2}\right)$$

Therefore: $f_1(t) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt \Rightarrow$

$$f_2(t) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos n(t+\pi) + b_n \sin n(t+\pi) =$$

$$= a_0 + \sum_{n=1}^{\infty} (-1)^n a_n \cos nt + (-1)^n b_n \sin nt$$

Fourier series

$$f_3(t) \sim \left(a_0 - \frac{\pi}{2}\right) + \sum_{n=1}^{\infty} a_n \cos\left(nt + \frac{n\pi}{2}\right) +$$

$$+ b_n \sin\left(nt + \frac{n\pi}{2}\right) =$$

$$= \alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos nt + \beta_n \sin nt \quad , \quad \text{where}$$

$$\alpha_0 = a_0 - \frac{\pi}{2}$$

$$\alpha_n = \begin{cases} a_n, & n = 4l \\ -a_n, & n = 4l+2 \\ b_n, & n = 4l+1 \\ -b_n, & n = 4l+3 \end{cases} \qquad \text{for some integer } l$$

$$\beta_n = \begin{cases} b_n & n = 4l \\ -b_n & n = 4l+2 \\ -a_n & n = 4l+1 \\ a_n & n = 4l+3 \end{cases}$$

(#4) Cos-coefficients of an odd functions are all 0. Nothing to calculate.

(#5) a. $f(t) = t^3 \quad 0 < t < \pi$

Cos-Fourier series:

$$a_0 = \frac{1}{\pi}\int_0^\pi t^3\,dt = \frac{\pi^3}{4}$$

$$a_n = \frac{2}{\pi}\int_0^\pi t^3 \cos nt\,dt = (-1)^n \frac{6\pi}{n^2} + \frac{12}{\pi n^4}\left((-1)^n - 1\right)$$

Sin-Fourier series:

$$b_n = \frac{2}{\pi}\int_0^\pi t^3 \sin nt\,dt = (-1)^{n+1}\left(\frac{2\pi^2}{n} - \frac{12}{n^3}\right)$$

These formulas com just from integration by parts.

(b) $f(t) = \cos t \quad 0 < t < \pi$

Cos. Fourier series:

$$f(t) = \cos t$$

Sin. Fourier series:

$$b_n = \frac{2}{\pi}\int_0^{\pi} \cos t \sin nt \, dt =$$

$$= \frac{1}{\pi}\left((-1)^{n+1} - 1\right)\frac{2n}{n^2-1}$$

(#6)

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\pi t + b_n \sin n\pi t)$$

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(t)\, dt$$

$$a_n = \int_{-1}^{1} f(t) \cos n\pi t \, dt, \quad b_n = \int_{-1}^{1} f(t) \sin n\pi t \, dt, \quad n \geq 1$$

Of course you can integrate over any segment of length 2.

#7

EXAMPLE 2 Let $f(t)$ be a function of period 2 with $f(t) = t^2$ if $0 < t < 2$. We define $f(t)$ for t an even integer by the average value condition in (13); consequently, $f(t) = 2$ if t is an even integer. The graph of the function f appears in Fig. 9.2.3. Find its Fourier series.

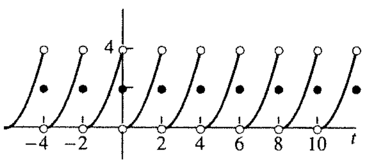


FIGURE 9.2.3. The period 2 function of Example 2

Solution Here $L = 1$, and it is most convenient to integrate from $t = 0$ to $t = 2$. Then

$$a_0 = \frac{1}{1}\int_0^2 t^2\,dt = \left[\frac{1}{3}t^3\right]_0^2 = \frac{8}{3}.$$

With the aid of the integral formulas in Eqs. (22) through (25) of Section 9.1 we obtain

$$\begin{aligned} a_n &= \int_0^2 t^2 \cos n\pi t\,dt \\ &= \frac{1}{n^3\pi^3}\int_0^{2n\pi} u^2 \cos u\,du \qquad \left(u = n\pi t,\ t = \frac{u}{n\pi}\right) \\ &= \frac{1}{n^3\pi^3}\Big[u^2 \sin u - 2\sin u + 2u\cos u\Big]_0^{2n\pi} = \frac{4}{n^2\pi^2}; \end{aligned}$$

$$\begin{aligned} b_n &= \int_0^2 t^2 \sin n\pi t\,dt = \frac{1}{n^3\pi^3}\int_0^{2n\pi} u^2 \sin u\,du \\ &= \frac{1}{n^3\pi^3}\Big[-u^2 \cos u + 2\cos u + 2u\sin u\Big]_0^{2n\pi} = -\frac{4}{n\pi}. \end{aligned}$$

Hence the Fourier series of f is

$$f(t) = \frac{4}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos n\pi t}{n^2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi t}{n}, \tag{15}$$

and Theorem 1 assures us that this series converges to $f(t)$ for all t. ■

We can draw some interesting consequences from the Fourier series in (15). If we substitute $t = 0$ on each side, we find that

$$f(0) = 2 = \frac{4}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

On solving for the series, we obtain the lovely summation

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6} \tag{16}$$

that was discovered by Euler. If we substitute $t = 1$ in Eq. (15), we get

$$f(1) = 1 = \frac{4}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2},$$

which yields

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}. \tag{17}$$

If we add the series in Eqs. (16) and (17) and then divide by 2, the "even" terms cancel and the result is

$$\sum_{n\text{ odd}}\frac{1}{n^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}. \tag{18}$$

Fourier series

\#8

$$f(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\pi t}$$

$$c_n = \frac{1}{2} \int_{-1}^{1} f(t) e^{-in\pi t} dt$$

$$f(t) = e^t \qquad -1 < t < 1 \qquad \Rightarrow$$

$$\Rightarrow \quad c_n = \frac{1}{2} \int_{-1}^{1} e^{t(1-in\pi)} dt =$$

$$= \frac{1}{2} \frac{1}{1-in\pi} e^{t(1-in\pi)} \Big|_{t=-1}^{1} =$$

$$= \frac{(-1)^n}{1-in\pi} \operatorname{sh} 1$$

$$e^t \sim \operatorname{sh} 1 \sum_{-\infty}^{\infty} \frac{(-1)^n}{1-in\pi} e^{int}$$

#9

EXAMPLE 1

Find the temperature $u(x, t)$ at any time in a metal rod 50 cm long, insulated on the sides, which initially has a uniform temperature of 20°C throughout, and whose ends are maintained at 0°C for all $t > 0$.

The temperature in the rod satisfies the heat conduction problem (1), (3), (4) with $L = 50$ and $f(x) = 20$ for $0 < x < 50$. Thus, from Eq. (19), the solution is

$$u(x, t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/2500} \sin \frac{n\pi x}{50}, \tag{22}$$

where, from Eq. (21),

$$\begin{aligned} c_n &= \frac{4}{5} \int_0^{50} \sin \frac{n\pi x}{50}\, dx \\ &= \frac{40}{n\pi}(1 - \cos n\pi) = \begin{cases} 80/n\pi, & n \text{ odd;} \\ 0, & n \text{ even.} \end{cases} \end{aligned} \tag{23}$$

Finally, by substituting for c_n in Eq. (22) we obtain

$$u(x, t) = \frac{80}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} e^{-n^2\pi^2\alpha^2 t/2500} \sin \frac{n\pi x}{50}. \tag{24}$$

The expression (24) for the temperature is moderately complicated, but the negative exponential factor in each term of the series causes the series to converge quite rapidly,

See next page

except for small values of t or α^2. Therefore accurate results can usually be obtained by using only a few terms of the series.

In order to display quantitative results let us measure t in seconds; then α^2 has the units of cm^2/sec. If we choose $\alpha^2 = 1$ for convenience, this corresponds to a rod of a material whose thermal properties are somewhere between copper and aluminum. The behavior of the solution can be seen from the graphs in Figures 10.5.3 through 10.5.5. In Figure 10.5.3 we show the temperature distribution in the bar at several different times. Observe that the temperature diminishes steadily as heat in the bar is lost through the endpoints. The way in which the temperature decays at a given point in the bar is indicated in Figure 10.5.4, where temperature is plotted against time for a few selected points in the bar. Finally, Figure 10.5.5 is a three-dimensional plot of u versus both x and t. Observe that the graphs in Figures 10.5.3 and 10.5.4 are obtained by intersecting the surface in Figure 10.5.5 by planes on which either t or x is constant. The slight waviness in Figure 10.5.5 at $t = 0$ results from using only a finite number of terms in the series for $u(x, t)$ and from the slow convergence of the series for $t = 0$.

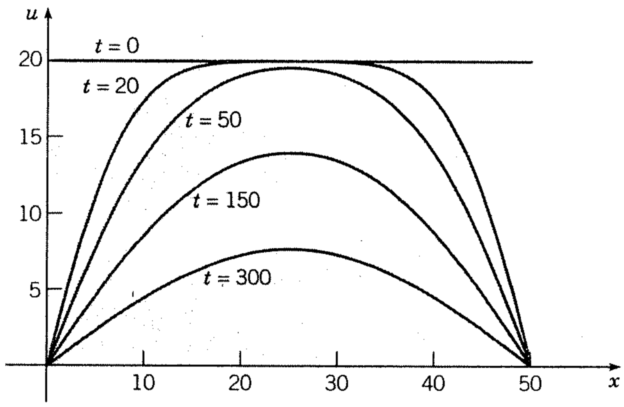


FIGURE 10.5.3 Temperature distributions at several times for the heat conduction problem of Example 1.

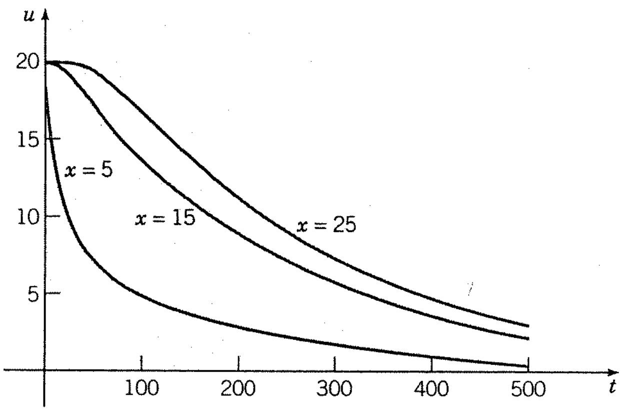


FIGURE 10.5.4 Dependence of temperature on time at several locations for the heat conduction problem of Example 1.

See next page

Fourier series

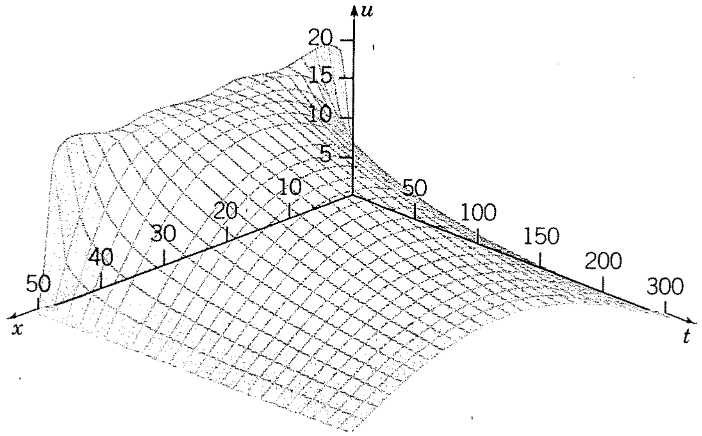


FIGURE 10.5.5 Plot of temperature u versus x and t for the heat conduction problem of Example 1.

A question with possible practical implications is to determine the time τ at which the entire bar has cooled to a specified temperature. For example, when is the temperature in the entire bar no greater than 1°C? Because of the symmetry of the initial temperature distribution and the boundary conditions, the warmest point in the bar is always the center. Thus, τ is found by solving $u(25, t) = 1$ for t. Using one term in the series expansion (24), we obtain

$$\tau = \frac{2500}{\pi^2} \ln(80/\pi) \cong 820 \text{ sec.}$$

Comment. You are not expected to make numerical simulation. I included this materials just because I have a cheap possibility to show you the whole picture

Fourier series

#10

EXAMPLE 1

Consider the heat conduction problem

$$u_{xx} = u_t, \quad 0 < x < 30, \quad t > 0, \tag{18}$$

$$u(0,t) = 20, \quad u(30,t) = 50, \quad t > 0, \tag{19}$$

$$u(x,0) = 60 - 2x, \quad 0 < x < 30. \tag{20}$$

Find the steady-state temperature distribution and the boundary value problem that determines the transient distribution.

The steady-state temperature satisfies $v''(x) = 0$ and the boundary conditions $v(0) = 20$ and $v(30) = 50$. Thus $v(x) = 20 + x$. The transient distribution $w(x,t)$ satisfies the heat conduction equation

$$w_{xx} = w_t, \tag{21}$$

the homogeneous boundary conditions

$$w(0,t) = 0, \quad w(30,t) = 0, \tag{22}$$

and the modified initial condition

$$w(x,0) = 60 - 2x - (20 + x) = 40 - 3x. \tag{23}$$

Note that this a standard problem with $f(x) = 40 - 3x$, $\alpha^2 = 1$, and $L = 30$.

Figure 10.6.1 shows a plot of the initial temperature distribution $60 - 2x$, the final temperature distribution $20 + x$, and the temperature at two intermediate times found by solving Eqs. (21) through (23). Note that the intermediate temperature satisfies the boundary conditions (19) for any $t > 0$. As t increases, the effect of the boundary conditions gradually moves from the ends of the bar toward its center.

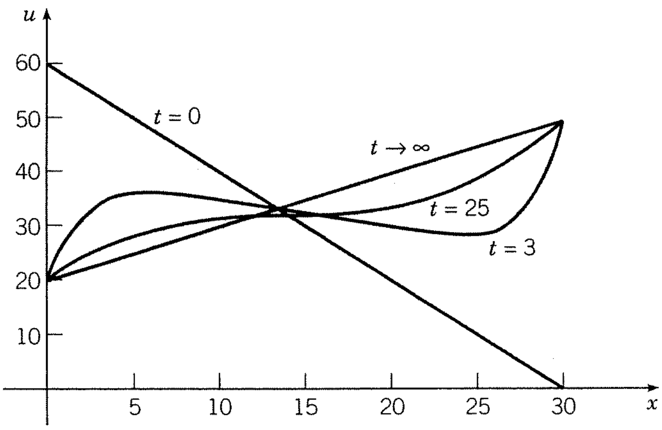


FIGURE 10.6.1 Temperature distributions at several times for the heat conduction problem of Example 1.

I skip solving the standard problem.

#11

EXAMPLE 1

Consider a vibrating string of length $L = 30$ that satisfies the wave equation

$$4u_{xx} = u_{tt}, \quad 0 < x < 30, \quad t > 0 \tag{23}$$

Assume that the ends of the string are fixed, and that the string is set in motion with no initial velocity from the initial position

$$u(x,0) = f(x) = \begin{cases} x/10, & 0 \le x \le 10, \\ (30-x)/20, & 10 < x \le 30. \end{cases} \tag{24}$$

Find the displacement $u(x,t)$ of the string and describe its motion through one period.

The solution is given by Eq. (20) with $a = 2$ and $L = 30$, that is,

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{30}\cos\frac{2n\pi t}{30}, \tag{25}$$

where c_n is calculated from Eq. (22). Substituting from Eq. (24) into Eq. (22), we obtain

$$c_n = \frac{2}{30}\int_0^{10} \frac{x}{10}\sin\frac{n\pi x}{30}\,dx + \frac{2}{30}\int_{10}^{30} \frac{30-x}{20}\sin\frac{n\pi x}{30}\,dx. \tag{26}$$

By evaluating the integrals in Eq. (26), we find that

$$c_n = \frac{9}{n^2\pi^2}\sin\frac{n\pi}{3}, \quad n = 1, 2, \ldots. \tag{27}$$

The solution (25), (27) gives the displacement of the string at any point x at any time t. The motion is periodic in time with period 30, so it is sufficient to analyze the solution for $0 \le t \le 30$.

The best way to visualize the solution is by a computer animation showing the dynamic behavior of the vibrating string. Here we indicate the motion of the string in Figures 10.7.4, 10.7.5, and 10.7.6. Plots of u versus x for $t = 0, 4, 7.5, 11$, and 15 are shown in Figure 10.7.4. Observe that the maximum initial displacement is positive and occurs at $x = 10$, while at $t = 15$, a half period later, the maximum displacement is negative and occurs at $x = 20$. The string then retraces its motion and returns to its original configuration at $t = 30$. Figure 10.7.5 shows the behavior of the points $x = 10, 15$, and 20 by plots of u versus t for these fixed values of x. The plots confirm that the motion is indeed periodic with period 30. Observe also that each interior point

See next page

on the string is motionless for one-third of each period. Figure 10.7.6 shows a three-dimensional plot of u versus both x and t, from which the overall nature of the solution is apparent. Of course, the curves in Figures 10.7.4 and 10.7.5 lie on the surface shown in Figure 10.7.6.

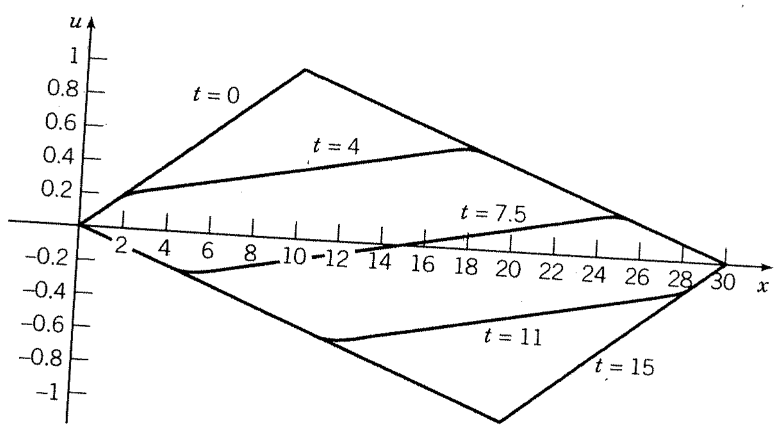


FIGURE 10.7.4 Plots of u versus x for fixed values of t for the string in Example 1.

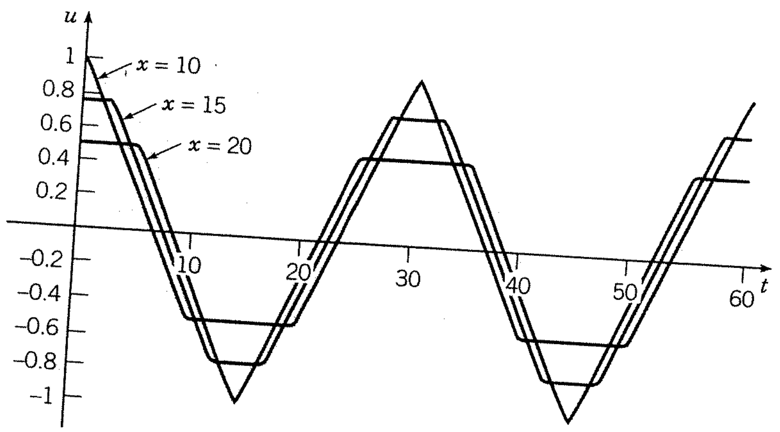


FIGURE 10.7.5 Plots of u versus t for fixed values of x for the string in Example 1.