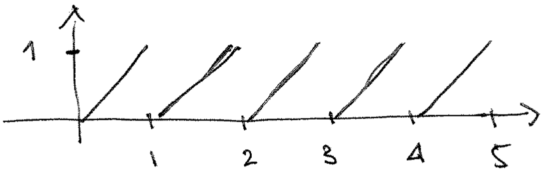
Project for weeks 9 and 10

Laplace transform

Find the Laplace transforms:

1. $\mathcal{L}(5 - 3t + 4\sin 2t - 6e^{4t})$.
2. $\mathcal{L}(e^{-3t}\sin 2t)$
3. $\mathcal{L}(t^2 e^t)$
4. $$\mathcal{L}\left\{\int_0^t (\tau^3 + \sin 2\tau)d\tau\right\}$$
5. The *gamma function* is defined by
$$\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt, \quad (\alpha > 0).$$
- prove that $\Gamma(1) = 1$,
- prove that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$. In fact, if n is a positive integer, show that $\Gamma(n+1) = n!$
6. Find $\mathcal{L}(f)$, where f is defined by
$$f(t) = \begin{cases} 3, & \text{for } 0 \le t < 4; \\ -5, & \text{for } 4 \le t < 6; \\ e^{-t}, & \text{for } t > 6. \end{cases}$$
7. Find $\mathcal{L}(f)$, where f is defined by
$$f(t) = \begin{cases} t, & \text{if } 0 \le t < 1; \\ -2-t, & \text{if } 1 \le t < 3; \\ t-4, & \text{if } 3 \le t < 46; \\ 0, & \text{otherwise.} \end{cases}$$
Sketch the graph of f.
8. Find $\mathcal{L}(f)$, where f is defined by
$$f(t) = [u(t-1) - u(t-3)]\sin 3t,$$
here u is the Heaviside function. Sketch the graph of f.

9. The following image shows the graph of a periodic function $f(t)$.



Find $\mathcal{L}(f)$

10. In mathematics the symbol $[t]$ calls for the greatest integer not exeeding the number t. For example $[3\frac{1}{2}] = 3$ and $[-1.2] = -2$. Define $f(t) = [t]$ for $t > 0$.
 - sketch the graph of f. Remark why this function is often called the *staircas function*
 - find $\mathcal{L}(f)$. (Hint. Use the results of the previous item).

Find the inverse Laplace transforms:

1. $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2(s^2+9)}\right\}$
2. $\mathcal{L}^{-1}\left\{\frac{s+7}{s^2+2s+5}\right\}$
3. Find $\mathcal{L}^{-1}(F)$ where
$$F(s) = \frac{e^{-2s}}{s(s^2+9)}$$
4. Find $\mathcal{L}^{-1}(e^{2s})$
5. Let $f(t) = \sin t$ and $g(t) = t$, the both are defined for $t > 0$. Compute the convolution $f * g$ in two ways. First directly from the definition of the convolution. Next, compute $F = \mathcal{L}(f)$ and $G = \mathcal{L}(g)$, and then compute $f * g = \mathcal{L}^{-1}(FG)$. Take care about obtaining the same result!

Solve the initial value problems.

1. $$y'' + 2y' + 2y = \cos 2t, \text{ for } t > 0 \quad \text{with } y(0) = 0 \text{ and } y'(0) = 1.$$
2. $$x''(t) - 2x'(t) - 3x(t) = e^{2t}, \text{ for } t > 3 \quad \text{with } x(3) = 1 \text{ and } x'(3) = 0.$$
3. $$y'' - 4y' + 3y = \delta(t), \text{ for } t > 0 \quad \text{with } y(0) = 0 \text{ and } y'(0) = 0,$$
 here δ denotes the Dirac delta function.

4. Prove that the solution to the equation

$$y'' + y = g(t), \text{ for } t > 0 \quad \text{with } y(0) = 1 \text{ and } y'(0) = -2,$$

where g is, say, some piecewise continuous bounded function, has the form

$$y(t) = \cos t - 2\sin t + \int_0^t \sin(t-\tau)g(\tau)d\tau$$

5. Solve the system of differential equations:

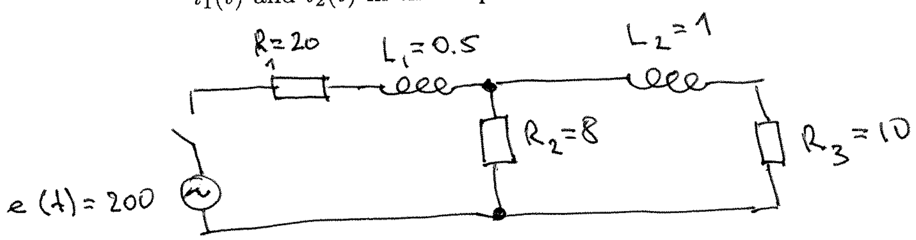
$$y_1'(t) + y_2'(t) + 5y_1(t) + 3y_2(t) = e^{-t}$$

$$2y_1'(t) + y_2'(t) + Y_1(t) + y_2(t) = 3$$

provided that $y_1(0) = 2$ and $y_2(0) = 1$

Applications

1. In the parallel network on the picture belowthere is no current flowing in either loop prior to closing the switch at time $t = 0$. Deduce the currents $i_1(t)$ and $i_2(t)$ in the loops at the time t.

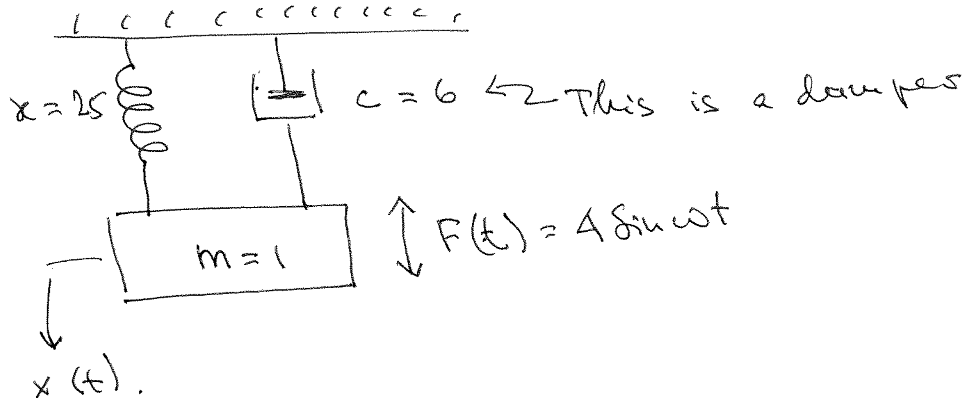


2. The mass of the mass-spring-damper sysstem on the figure below is subject to an externally applied periodic force $F(t) = 4\sin\omega t$ starting from the moment $t = 0$. Determine the resulting displacement of the mass at time t, given that $x(0) = 0$ and $x'(0) = 0$, for the two cases:

(a) $\omega = 2$ (b) $\omega = 5$

In the case $\omega = 5$, what would happen to the response if the damper were missiing?

Figure to problem 2 :



This is the Laplace part of the project.

The rest will be published soon.