Laplace transform

EXAMPLE 11.8 Determine $\mathscr{L}\{5 - 3t + 4\sin 2t - 6e^{4t}\}$.

Solution Using the results given in (11.4)–(11.7),

#1

$$\mathscr{L}\{5\} = \frac{5}{s}, \quad \mathrm{Re}(s) > 0 \qquad \mathscr{L}\{t\} = \frac{1}{s^2}, \quad \mathrm{Re}(s) > 0$$

$$\mathscr{L}\{\sin 2t\} = \frac{2}{s^2 + 4}, \quad \mathrm{Re}(s) > 0 \qquad \mathscr{L}\{e^{4t}\} = \frac{1}{s - 4}, \quad \mathrm{Re}(s) > 4$$

so, by the linearity property,

$$\mathscr{L}\{5 - 3t + 4\sin 2t - 6e^{4t}\} = \mathscr{L}\{5\} - 3\mathscr{L}\{t\} + 4\mathscr{L}\{\sin 2t\} - 6\mathscr{L}\{e^{4t}\}$$

$$= \frac{5}{s} - \frac{3}{s^2} + \frac{8}{s^2 + 4} - \frac{6}{s - 4}, \quad \mathrm{Re}(s) > \max\{0, 4\}$$

$$= \frac{5}{s} - \frac{3}{s^2} + \frac{8}{s^2 + 4} - \frac{6}{s - 4}, \quad \mathrm{Re}(s) > 4$$

EXAMPLE 11.10 Determine $\mathscr{L}\{e^{-3t}\sin 2t\}$.

Solution From the result (11.7),

$$\mathscr{L}\{\sin 2t\} = F(s) = \frac{2}{s^2 + 4}, \quad \mathrm{Re}(s) > 0$$

#2

so, by the first shift theorem,

$$\mathscr{L}\{e^{-3t}\sin 2t\} = F(s + 3) = [F(s)]_{s \to s+3}, \quad \mathrm{Re}(s) > 0 - 3$$

that is,

$$\mathscr{L}\{e^{-3t}\sin 2t\} = \frac{2}{(s + 3)^2 + 4} = \frac{2}{s^2 + 6s + 13}, \quad \mathrm{Re}(s) > -3$$

The function $e^{-3t}\sin 2t$ in Example 11.10 is a member of a general class of functions called **damped sinusoids**. These play an important role in the study of engineering systems, particularly in the analysis of vibrations.

EXAMPLE 11.12 Determine $\mathscr{L}\{t^2 e^t\}$.

Solution From the result (11.6),

$$\mathscr{L}\{e^t\} = F(s) = \frac{1}{s-1}, \quad \mathrm{Re}(s) > 1$$

#3

so, by the derivative theorem,

$$\mathscr{L}\{t^2 e^t\} = (-1)^2 \frac{d^2 F(s)}{ds^2} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s-1}\right)$$

$$= (-1)\frac{d}{ds}\left(\frac{1}{(s-1)^2}\right)$$

$$= \frac{2}{(s-1)^3}, \quad \mathrm{Re}(s) > 1$$

\# 4 $\mathcal{L}\left\{\int_0^t (\tau^3 + \sin 2\tau)\, d\tau\right\} =$

$$= \frac{1}{s}\mathcal{L}(\tau^3 + \sin 2\tau) = \frac{1}{s}\left(\frac{6}{s^4} + \frac{2}{s^2+4}\right) =$$

$$= \frac{6}{s^5} + \frac{2}{s(s^2+4)}$$

\# 5 - $\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1$ - explicit calculation

- $\Gamma(\alpha+1) = \int_0^\infty e^{-t} t^\alpha\, dt$. integration by parts:

$$= -\int_0^\infty t^\alpha\, d e^{-t} = \underbrace{-e^{-t} t^\alpha \Big|_0^\infty}_{=0} + \alpha \int_0^\infty e^{-t} t^{\alpha-1}\, dt =$$

$$= \alpha\,\Gamma(\alpha)$$

- $\Gamma(n+1) = n\,\Gamma(n) = n(n-1)\,\Gamma(n-1) = \cdots$

$$\cdots = n(n-1)\cdots 2 \cdot \Gamma(1) = n!$$

Laplace transform

6

$$f(t) = 3[u(t) - u(t-4)] -$$

$$- 5[u(t-4) - u(t-6)] + e^{-t} u(t-6) =$$

$$= 3u(t) - 8u(t-4) + 5u(t-6) + e^{-6} e^{-(t-6)} u(t-6)$$

$$\left.\begin{aligned} &\mathcal{L}(u(t-a)) = \frac{1}{s} e^{-as} \\ &\mathcal{L}(f(t-a)\,u(t-a)) = e^{-as} F(s) \end{aligned}\right\} - \text{general formulas}$$

$\Rightarrow$ in our case

$$\mathcal{L}(f) = \frac{1}{s}\left[3 - 8e^{-4s} + 5e^{-6s}\right] + e^{-6} \frac{e^{-6s}}{s+1}$$

7

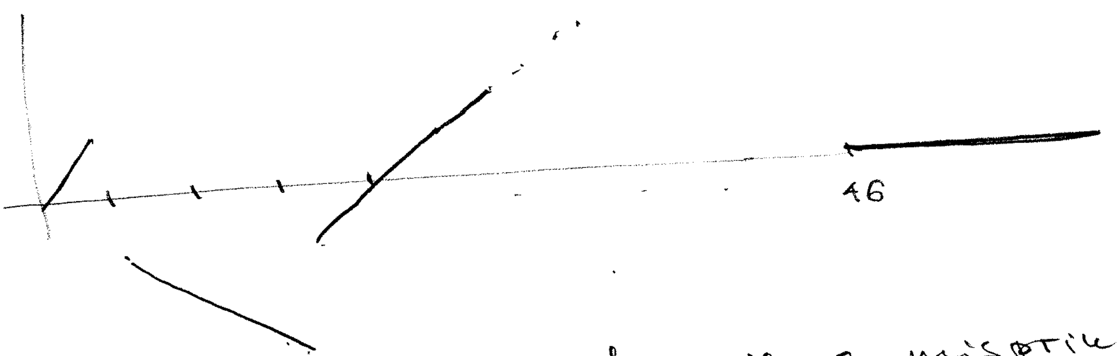
$$f(t) = t\left(u(t) - u(t-1)\right) -$$
$$-(2+t)\left(u(t-3) - u(t-1)\right) +$$
$$+ (t-4)\left(u(t-46) - u(t-3)\right) =$$

$$= t - \cancel{(t-1)u(t-1)} - u(t-1) +$$
$$+ \cancel{(t-1)u(t-1)} + 3u(t-1) -$$
$$- (t-3)u(t-3) - 5u(t-3) -$$
$$- (t-3)u(t-3) - u(t-3) +$$
$$+ (t-46)u(t-46) + 40\,u(t-46) =$$

$$= t + 2u(t-1) - 6u(t-3) - 2(t-3)u(t-3) +$$
$$+ 40\,u(t-46) + (t-46)u(t-46)$$

$$\Rightarrow \mathcal{L}(f) = \frac{1}{s^2} + \frac{2}{s}e^{-s} - \frac{6}{s}e^{-3s} - \frac{2}{s^2}e^{-3s} +$$
$$+ \frac{40}{s}e^{-46s} + \frac{1}{s^2}e^{-46s}$$

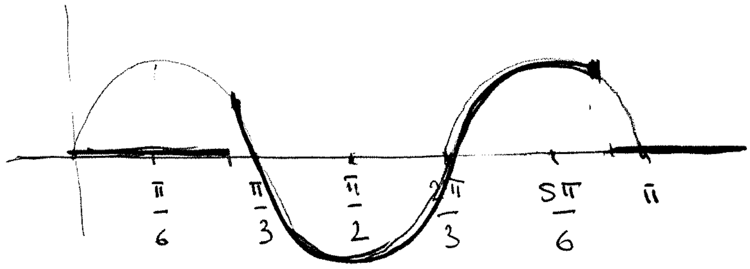
<u>Graph</u>:



There is a misprint.
I ment 4, not 46.

Anyway

#8



Bold line gives the graph.

$$f(t) = \sin 3t \; u(t-1) - \sin 3t \; u(t-3) =$$

$$= \sin(3(t-1)+3)\, u(t-1) - \sin(3(t-3)+9)\, u(t-3) =$$

$$= \cos 3 \cdot \sin 3(t-1)\, u(t-1) + \sin 3 \cos 3(t-1)\, u(t-1) -$$

$$- \cos 9 \; \sin 3(t-3)\, u(t-3) - \sin 9 \; \cos 3(t-3)\, u(t-3)$$

$\Rightarrow$

$$\mathcal{L}(f) = \cos 3 \cdot \frac{e^{-s}}{s^2+1} + \sin 3 \; \frac{s e^{-s}}{s^2+1} -$$

$$-3\cos 9 \; \frac{e^{-3s}}{s^2+9} - \sin 9 \; \frac{s e^{-3s}}{s^2+9}$$

Laplace transform

#9 We have $f(t) = t-n$ for $n<t<n+1$

Therefore

$$\int_0^\infty f(t)\, e^{-ts}\, dt = \sum_{n=0}^{\infty} \int_n^{n+1} f(t)\, e^{-ts}\, dt =$$

$$= \sum_{n=0}^{\infty} \int_n^{n+1} (t-n)\, e^{-(t-n)s} \cdot e^{-ns}\, dt =$$

$$= \sum_{n=0}^{\infty} e^{-ns} \int_0^1 \tau\, e^{-\tau s}\, d\tau =$$

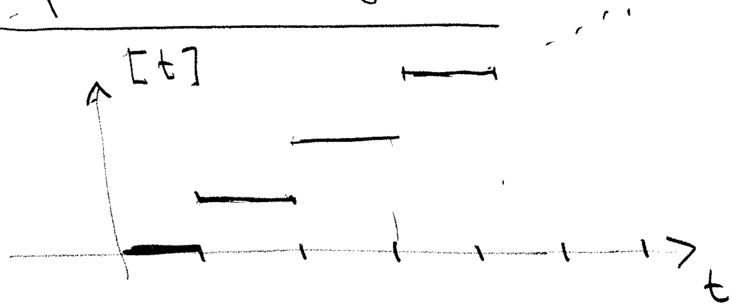
$$= \underbrace{\left(\int_0^1 \tau\, e^{-\tau s}\, d\tau\right)}_{-\frac{1}{s} e^{-s} - \frac{1}{s^2}(e^{-s}-1)} \underbrace{\left(\sum_{n=0}^{\infty} e^{-ns}\right)}_{= \frac{1}{1-e^{-s}}}$$

Finally

$$\mathcal{L}(f) = \frac{1}{s^2} + \frac{1}{s} \frac{e^{-s}}{e^{-s}-1}$$

Laplace transform

#10



this is the graph.

In order to find the Laplace transform of $[t]$ just mention that

$$f(t) = t - [t] ,$$

where f is the function from problem # 9.

Therefore

$$\mathcal{L}(f) - \mathcal{L}(t) = \mathcal{L}([t])$$

and you can easily find the left-hand side.

EXAMPLE 11.17 Find

#1

$$\mathscr{L}^{-1}\left\{\frac{s+1}{s^2(s^2+9)}\right\}$$

Solution Resolving $(s+1)/s^2(s^2+9)$ into partial fractions gives

$$\frac{s+1}{s^2(s^2+9)} = \frac{\frac{1}{9}}{s} + \frac{\frac{1}{9}}{s^2} - \frac{1}{9}\frac{s+1}{s^2+9}$$

$$= \frac{\frac{1}{9}}{s} + \frac{\frac{1}{9}}{s^2} - \frac{1}{9}\frac{s}{s^2+3^2} - \frac{1}{27}\frac{3}{s^2+3^2}$$

Using the results in Figure 11.5, together with the linearity property, we have

$$\mathscr{L}^{-1}\left\{\frac{s+1}{s^2(s^2+9)}\right\} = \tfrac{1}{9} + \tfrac{1}{9}t - \tfrac{1}{9}\cos 3t - \tfrac{1}{27}\sin 3t$$

EXAMPLE 11.20 Find

$$\mathscr{L}^{-1}\left\{\frac{s+7}{s^2+2s+5}\right\}$$

Solution

#2

$$\frac{s+7}{s^2+2s+5} = \frac{s+7}{(s+1)^2+4} = \frac{(s+1)}{(s+1)^2+4} + 3\frac{2}{(s+1)^2+4}$$

$$= \left[\frac{s}{s^2+2^2}\right]_{s\to s+1} + 3\left[\frac{2}{s^2+2^2}\right]_{s\to s+1}$$

Since $s/(s^2+2^2) = \mathscr{L}\{\cos 2t\}$ and $2/(s^2+2^2) = \mathscr{L}\{\sin 2t\}$, the shift theorem gives

$$\mathscr{L}^{-1}\left\{\frac{s+7}{s^2+2s+5}\right\} = \mathrm{e}^{-t}\cos 2t + 3\mathrm{e}^{-t}\sin 2t$$

#3 $\frac{1}{s(s^2+9)} = \frac{1}{9}\left(\frac{1}{s} + \frac{s}{s^2+9}\right) \Rightarrow$

$\Rightarrow \mathcal{L}^{-1}(F) = \frac{1}{9}\, u(t-2) + \frac{1}{9}\cos 3(t-2)\, u(t-2)$

#4 $\mathcal{L}^{-1}(e^{2s}) =$ — sorry, this is a misprint.

Test: how you can see this?

#5

$(f*g)(t) = \int_0^t (t-\tau)\sin\tau\, d\tau =$

$= t\int_0^t \sin\tau\, d\tau - \int_0^t \tau\sin\tau\, d\tau =$

$= -t(\cos t - 1) + \tau\cos\tau\Big|_0^t - \int_0^t \cos\tau\, d\tau =$

$= -t\cos t + t + t\cos t - \sin t = t - \sin t$

$\mathcal{L}f = \frac{1}{s^2+1}$ $\mathcal{L}g = \frac{1}{s^2}$ $\mathcal{L}f\cdot\mathcal{L}g = \frac{1}{s^2(s^2+1)} =$

$= \frac{1}{s^2} - \frac{1}{s^2+1} \Rightarrow \mathcal{L}^{-1}(\mathcal{L}f\cdot\mathcal{L}g) = t - \sin t$

EXAMPLE 4.3 ◆ Use the Laplace transform to solve the initial value problem

$$y'' + 2y' + 2y = \cos 2t \quad \text{with} \quad y(0) = 0 \quad \text{and} \quad y'(0) = 1.$$

#1

We compute that

$$\mathcal{L}(y'' + 2y' + 2y) = s^2 Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + 2Y(s),$$

and

$$\mathcal{L}(\cos 2t) = \frac{s}{s^2+4}.$$

Setting these two expressions equal and substituting the initial conditions, we get

$$(s^2 + 2s + 2)Y(s) - 1 = \frac{s}{s^2+4},$$

or

$$Y(s) = \frac{1}{s^2+2s+2} + \frac{s}{(s^2+2s+2)(s^2+4)}. \tag{4.4}$$

The inverse of the Laplace transform of the first term on the right can be computed easily, so let's deal with the second term. Since $s^2 + 2s + 2 = (s+1)^2 + 1$ cannot be factored, the partial fractions decomposition of this term has the form

$$\frac{s}{(s^2+2s+2)(s^2+4)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+4}. \tag{4.5}$$

If we combine the terms on the right of (4.5) and multiply it out completely, we find that the numerators are

$$s = (A+C)s^3 + (B+2C+D)s^2 + (4A+2C+2D)s + (4B+2D).$$

Equating the coefficients of the powers, we get four equations:

$$\begin{aligned} A + \quad C &= 0, \\ B + 2C + \quad D &= 0, \\ 4A + 2C + 2D &= 1, \\ 4B + 2D &= 0. \end{aligned}$$

Solving this system, we get $A = 1/10$, $B = -1/5$, $C = -1/10$, and $D = 2/5$. Thus the second term on the right in (4.4) is

$$\frac{1}{10}\frac{s-2}{s^2+2s+2} - \frac{1}{10}\frac{s-4}{s^2+4},$$

and (4.4) becomes

See next page

$$Y(s) = \frac{1}{s^2+2s+2} + \frac{1}{10}\frac{s-2}{s^2+2s+2} - \frac{1}{10}\frac{s-4}{s^2+4}.$$

To better match the entries in Table 1, we complete the square to get $s^2+2s+2 = (s+1)^2+1$ and then rewrite the above equation as

$$\begin{aligned} Y(s) = {} & \frac{1}{(s+1)^2+1} + \frac{1}{10}\frac{s+1}{(s+1)^2+1} \\ & - \frac{3}{10}\frac{1}{(s+1)^2+1} - \frac{1}{10}\frac{s}{s^2+4} + \frac{2}{10}\frac{2}{s^2+4}. \end{aligned}$$

We can read the inverse Laplace transforms of the summands directly from Table 1 to find that

$$\begin{aligned} y(t) &= e^{-t}\sin t + \frac{1}{10}e^{-t}\cos t - \frac{3}{10}e^{-t}\sin t - \frac{1}{10}\cos 2t + \frac{2}{10}\sin 2t \\ &= \frac{1}{10}\left\{e^{-t}(\cos t + 7\sin t) + 2\sin 2t - \cos 2t\right\}. \end{aligned}$$

◆

EXAMPLE 4.7 ◆ Use the Laplace transform to compute the solution to the initial value problem

$$x''(t) - 2x'(t) - 3x(t) = e^{2t} \quad \text{with} \quad x(3) = 1 \quad \text{and} \quad x'(3) = 0.$$

Define

$$y(t) = x(t+3).$$

Then $y'(t) = x'(t+3)$ and $y''(t) = x''(t+3)$, so the initial value problem becomes

$$y''(t) - 2y'(t) - 3y(t) = e^{2(t+3)} = e^6e^{2t}, \quad \text{with} \quad y(0) = 1, \quad \text{and} \quad y'(0) = 0.$$

Taking the Laplace transform of the left-hand side of the differential equation, and using Proposition 2.4 and the initial conditions, we obtain

$$\begin{aligned} \mathcal{L}\{y'' - 2y' - 3y\} &= \left(s^2Y(s) - y'(0) - sy(0)\right) - 2\left(sY(s) - y(0)\right) - 3Y(s) \\ &= (s^2 - 2s - 3)Y(s) - s + 2. \end{aligned}$$

Since $\mathcal{L}(e^{2t}) = 1/(s-2)$, the differential equation is transformed into

$$(s^2 - 2s - 3)Y(s) - s + 2 = e^6\frac{1}{s-2}.$$

We can factor $s^2 - 2s - 3 = (s-3)(s+1)$, so we have

$$Y(s) = \frac{s-2}{(s-3)(s+1)} + \frac{e^6}{(s-2)(s-3)(s+1)}.$$

To find y, we use partial fractions in the usual manner to obtain

$$Y(s) = \frac{e^6+1}{4}\frac{1}{s-3} - \frac{e^6}{3}\frac{1}{s-2} + \frac{e^6+9}{12}\frac{1}{s+1}.$$

Taking the inverse Laplace transform using Table 1, we get

$$y(t) = \frac{e^6+1}{4}e^{3t} - \frac{e^6}{3}e^{2t} + \frac{e^6+9}{12}e^{-t}.$$

\# 3 $\quad y'' - 4y' + 3y = \delta(t), \quad t > 0$

$y(0) = 0, \; y'(0) = 0$

$\Rightarrow Y(s)\,(s^2 - 4s + 3) = 1$

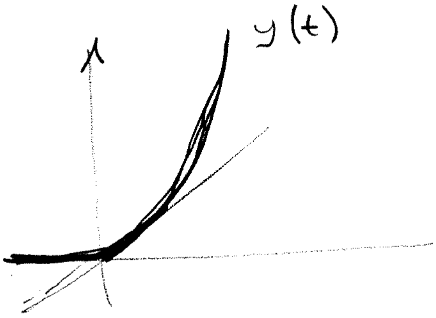
$$Y(s) = \frac{1}{s^2 - 4s + 3} = \frac{1}{2}\left(\frac{1}{s-3} - \frac{1}{s-1}\right) \Rightarrow$$

$$\Rightarrow y(t) = \frac{1}{2}\left(e^{3t} - e^{t}\right)$$

⚠ This function does NOT satisfy $y'(0) = 0$.

This is because y' is NOT continuous at $t = 0$

Here is the graph of the solution.



Nevertheless $y(t) \equiv 0$ for $t < 0$ so

$$\lim_{\varepsilon \to 0} y'(-\varepsilon) = 0,$$

i.e. $\quad y'(-0) = 0.$

\# 4. $y'' + y = g(t) \qquad y(0) = 1, \; y'(0) = -2$

$$\Rightarrow Y(s)(s^2+1) + 2 - s = G(s)$$

$$Y(s) = \frac{-2+s}{s^2+1} + \frac{G(s)}{s^2+1} \Rightarrow$$

$$y(t) = \mathcal{L}^{-1}\left(\frac{-2+s}{s^2+1}\right) + \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\, G(s)\right)$$

We have

$$\mathcal{L}^{-1}\left(\frac{-2+s}{s^2+1}\right) = \cos t - 2\sin t$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t \Rightarrow$$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\, G(s)\right) = \sin t * g$$

Finally

$$y(t) = \cos t - 2\sin t + \int_0^t \sin(t-\tau)\, g(\tau)\, d\tau.$$

Laplace transform

EXAMPLE 11.26 Solve for $t \geqslant 0$ the simultaneous first-order differential equations

#5

$$\frac{dx}{dt}+\frac{dy}{dt}+5x+3y=e^{-t} \tag{11.20}$$

$$2\frac{dx}{dt}+\frac{dy}{dt}+x+y=3 \tag{11.21}$$

subject to the initial conditions $x=2$ and $y=1$ at $t=0$.

Solution Taking Laplace transforms in (11.20) and (11.21) gives

$$sX(s)-x(0)+sY(s)-y(0)+5X(s)+3Y(s)=\frac{1}{s+1}$$

$$2[sX(s)-x(0)]+sY(s)-y(0)+X(s)+Y(s)=\frac{3}{s}$$

Rearranging and incorporating the given initial conditions $x(0)=2$ and $y(0)=1$ leads to

$$(s+5)X(s)+(s+3)Y(s)=3+\frac{1}{s+1}=\frac{3s+4}{s+1} \tag{11.22}$$

$$(2s+1)X(s)+(s+1)Y(s)=5+\frac{3}{s}=\frac{5s+3}{s} \tag{11.23}$$

Hence, by taking Laplace transforms, the pair of simultaneous differential equations (11.20) and (11.21) in $x(t)$ and $y(t)$ has been transformed into a pair of simultaneous algebraic equations (11.22) and (11.23) in the transformed variables $X(s)$ and $Y(s)$. These algebraic equations may now be solved simultaneously for $X(s)$ and $Y(s)$ using standard algebraic techniques.

Solving first for $X(s)$ gives

$$X(s)=\frac{2s^2+14s+9}{s(s+2)(s-1)}$$

Resolving into partial fractions,

$$X(s)=-\frac{\frac{9}{2}}{s}-\frac{\frac{11}{6}}{s+2}+\frac{\frac{25}{3}}{s-1}$$

See next page

which on inversion gives

$$x(t) = -\tfrac{9}{2} - \tfrac{11}{6}e^{-2t} + \tfrac{25}{3}e^{t} \quad (t \geqslant 0) \tag{11.24}$$

Likewise, solving for $Y(s)$ gives

$$Y(s) = \frac{s^3 - 22s^2 - 39s - 15}{s(s+1)(s+2)(s-1)}$$

Resolving into partial fractions,

$$Y(s) = \frac{\frac{15}{2}}{s} + \frac{\frac{1}{2}}{s+1} + \frac{\frac{11}{2}}{s+2} - \frac{\frac{25}{2}}{s-1}$$

which on inversion gives

$$y(t) = \tfrac{15}{2} + \tfrac{1}{2}e^{-t} + \tfrac{11}{2}e^{-2t} - \tfrac{25}{2}e^{t} \quad (t \geqslant 0)$$

Thus the solution to the given pair of simultaneous differential equations is

$$\left.\begin{aligned} x(t) &= -\tfrac{9}{2} - \tfrac{11}{6}e^{-2t} + \tfrac{25}{3}e^{t} \\ y(t) &= \tfrac{15}{2} + \tfrac{1}{2}e^{-t} + \tfrac{11}{2}e^{-2t} - \tfrac{25}{2}e^{t} \end{aligned}\right\} \quad (t \geqslant 0)$$

Note: When solving a pair of first-order simultaneous differential equations such as (11.20) and (11.21), an alternative approach to obtaining the value of $y(t)$ having obtained $x(t)$ is to use (11.20) and (11.21) directly.

Eliminating dy/dt from (11.20) and (11.21) gives

$$2y = \frac{dx}{dt} - 4x - 3 + e^{-t}$$

Substituting the solution obtained in (11.24) for $x(t)$ gives

$$2y = (\tfrac{11}{3}e^{-2t} + \tfrac{25}{3}e^{t}) - 4(-\tfrac{9}{2} - \tfrac{11}{6}e^{-2t} + \tfrac{25}{3})e^{t} - 3 + e^{-t}$$

leading as before to the solution

$$y = \tfrac{15}{2} + \tfrac{1}{2}e^{-t} + \tfrac{11}{2}e^{-2t} - \tfrac{25}{2}e^{t}$$

A further alternative is to express (11.22) and (11.23) in matrix form and solve for $X(s)$ and $Y(s)$ using Gaussian elimination.

In principle, the same procedure as used in Example 11.26 can be employed to solve a pair of higher-order simultaneous differential equations or a larger system of differential equations involving more unknowns. However, the algebra involved can become quite complicated, and matrix methods are usually preferred.

EXAMPLE 11.28 In the parallel network of Figure 11.9 there is no current flowing in either loop prior to closing the switch at time $t = 0$. Deduce the currents $i_1(t)$ and $i_2(t)$ flowing in the loops at time t.

#1

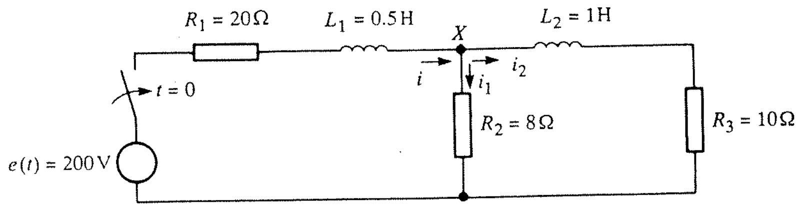


Figure 11.9 Parallel circuit of Example 11.28.

Solution Applying Kirchhoff's first law to node X gives

$$i = i_1 + i_2$$

Applying Kirchhoff's second law to each of the two loops in turn gives

$$R_1(i_1 + i_2) + L_1\frac{\mathrm{d}}{\mathrm{d}t}(i_1 + i_2) + R_2 i_1 = 200$$

$$L_2\frac{\mathrm{d}i_2}{\mathrm{d}t} + R_3 i_2 - R_2 i_1 = 0$$

See next page

Substituting the given values for the resistances and inductances gives

$$\left.\begin{aligned} &\frac{di_1}{dt} + \frac{di_2}{dt} + 56i_1 + 40i_2 = 400 \\ &\frac{di_2}{dt} - 8i_1 + 10i_2 = 0 \end{aligned}\right\} \tag{11.26}$$

Taking Laplace transforms and incorporating the initial conditions $i_1(0) = i_2(0) = 0$ leads to the transformed equations

$$(s + 56)I_1(s) + (s + 40)I_2(s) = \frac{400}{s} \tag{11.27}$$

$$-8I_1(s) + (s + 10)I_2(s) = 0 \tag{11.28}$$

Hence

$$I_2(s) = \frac{3200}{s(s^2 + 74s + 880)} = \frac{3200}{s(s + 59.1)(s + 14.9)}$$

Resolving into partial fractions gives

$$I_2(s) = \frac{3.64}{s} + \frac{1.22}{s + 59.1} - \frac{4.86}{s + 14.9}$$

which, on taking inverse transforms, leads to

$$i_2(t) = 3.64 + 1.22e^{-59.1t} - 4.86e^{-14.9t}$$

From (11.26),

$$i_1(t) = \frac{1}{8}\left(10i_2 + \frac{di_2}{dt}\right)$$

that is,

$$i_1(t) = 4.55 - 7.49e^{-59.1t} + 2.98e^{-14.9t}$$

Note that as $t \to \infty$, the currents $i_1(t)$ and $i_2(t)$ approach the constant values 4.55 and 3.64 A respectively. (Note that $i(0) = i_1(0) + i_2(0) \neq 0$ due to rounding errors in the calculation.)

EXAMPLE 11.30 The mass of the mass–spring–damper system of Figure 11.12(a) is subjected to an externally applied periodic force $F(t) = 4\sin\omega t$ at time $t = 0$. Determine the resulting displacement $x(t)$ of the mass at time t, given that $x(0) = \dot{x}(0) = 0$, for the two cases

\# 2

(a) $\omega = 2$ (b) $\omega = 5$

In the case $\omega = 5$, what would happen to the response if the damper were missing?

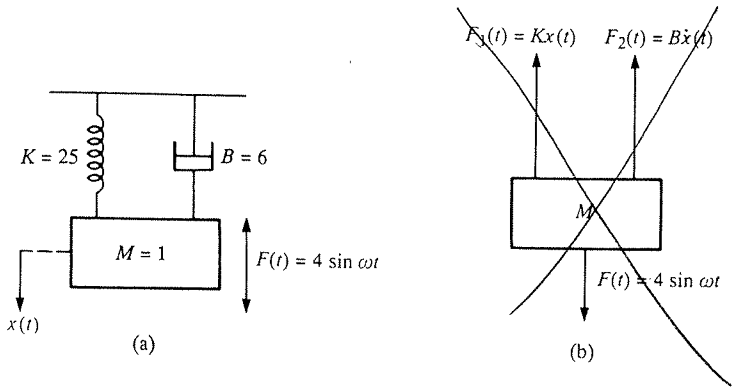


Figure 11.12 Mass–spring–damper system of Example 11.30.

Solution As indicated in Figure 11.12(b), the forces acting on the mass M are the applied forces $F(t)$ and the restoring forces F_1 and F_2 due to the spring and damper respectively. Thus, by Newton's law,

$$M\ddot{x}(t) = F(t) - F_1(t) - F_2(t)$$

Since $M = 1$, $F(t) = 4\sin\omega t$, $F_1(t) = Kx(t) = 25x(t)$ and $F_2(t) = B\dot{x}(t) = 6\dot{x}(t)$, this gives

$$\ddot{x}(t) + 6\dot{x}(t) + 25x(t) = 4\sin\omega t \quad \textbf{(11.31)}$$

as the differential equation representing the motion of the system.

Taking Laplace transforms throughout in (11.31) gives

$$(s^2 + 6s + 25)X(s) = [sx(0) + \dot{x}(0)] + 6x(0) + \frac{4\omega}{s^2 + \omega^2}$$

where $X(s)$ is the transform of $x(t)$. Incorporating the given initial conditions $x(0) = \dot{x}(0) = 0$ leads to

$$X(s) = \frac{4\omega}{(s^2 + \omega^2)(s^2 + 6s + 25)} \quad \textbf{(11.32)}$$

In case (a), with $\omega = 2$, (11.32) gives

$$X(s) = \frac{8}{(s^2 + 4)(s^2 + 6s + 25)}$$

which, on resolving into partial fractions, leads to

$$X(s) = \tfrac{4}{195}\frac{-4s + 14}{s^2 + 4} + \tfrac{2}{195}\frac{8s + 20}{s^2 + 6s + 25}$$

$$= \tfrac{4}{195}\frac{-4s + 14}{s^2 + 4} + \tfrac{2}{195}\frac{8(s + 3) - 4}{(s + 3)^2 + 16}$$

See next page

Taking inverse Laplace transforms gives the required response

$$x(t) = \tfrac{4}{195}(7\sin 2t - 4\cos 2t) + \tfrac{2}{195}e^{-3t}(8\cos 4t - \sin 4t) \qquad \textbf{(11.33)}$$

In case (b), with $\omega = 5$, (11.32) gives

$$X(s) = \frac{20}{(s^2+25)(s^2+6s+25)} \qquad \textbf{(11.34)}$$

that is,

$$X(s) = \frac{-\frac{2}{15}s}{s^2+25} + \frac{1}{15}\,\frac{2(s+3)+6}{(s+3)^2+16}$$

which, on taking inverse Laplace transforms, gives the required response

$$x(t) = -\tfrac{2}{15}\cos 5t + \tfrac{1}{15}e^{-3t}(2\cos 4t + \tfrac{3}{2}\sin 4t) \qquad \textbf{(11.35)}$$

If the damping term were missing then (11.34) would become

$$X(s) = \frac{20}{(s^2+25)^2} \qquad \textbf{(11.36)}$$

By Theorem 11.3,

$$\mathscr{L}\{t\cos 5t\} = -\frac{d}{ds}\mathscr{L}\{\cos 5t\} = -\frac{d}{ds}\left(\frac{s}{s^2+25}\right)$$

that is,

$$\mathscr{L}\{t\cos 5t\} = -\frac{1}{s^2+25} + \frac{2s^2}{(s^2+25)^2} = \frac{1}{s^2+25} - \frac{50}{(s^2+25)^2}$$

$$= \tfrac{1}{5}\mathscr{L}\{\sin 5t\} - \frac{50}{(s^2+25)^2}$$

Thus, by the linearity property (11.10),

$$\mathscr{L}\{\tfrac{1}{5}\sin 5t - t\cos 5t\} = \frac{50}{(s^2+25)^2}$$

so that taking inverse Laplace transforms in (11.36) gives the response as

$$x(t) = \tfrac{2}{25}(\sin 5t - 5t\cos 5t)$$

Because of the term $t\cos 5t$, the response $x(t)$ is unbounded as $t \to \infty$. This arises because in this case the applied force $F(t) = 4\sin 5t$ is in **resonance** with the system (that is, the vibrating mass), whose natural oscillating frequency is $5/2\pi$ Hz, equal to that of the applied force. Even in the presence of damping, the amplitude of the system response is maximized when the applied force is approaching resonance with the system. (This is left as an exercise for the reader.) In the absence of damping we have the limiting case of **pure resonance**, leading to an unbounded response. As noted in Chapter 10, Section 10.10.3, resonance is of practical importance, since, for example, it can lead to large and strong structures collapsing under what appears to be a relatively small force.