Numerical formulas

- Let $p(x)$ be the polynomial of degree $\leq n$ which coincides with $f(x)$ at points x_i, $i = 0, 1, \ldots, n$. Under the assumption that x and all the nodes x_i lie in the interval $[a, b]$, we have

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (x - x_i)$$

If the nodes are evenly spaced, including end points, and $|f^{(n+1)}(x)| \leq M$, we have

$$|f(x) - p(x)| \leq \frac{1}{4(n+1)} M \left(\frac{b-a}{n} \right)^{n+1}$$

- Numerical differentiation:

$$f'(x) = \frac{1}{h}(f(x+h) - f(x)) - \frac{h}{2} f''(\xi)$$
$$f'(x) = \frac{1}{h}(f(x) - f(x-h)) + \frac{h}{2} f''(\xi)$$
$$f'(x) = \frac{1}{2h}(f(x+h) - f(x-h)) - \frac{h^2}{6} f'''(\xi)$$
$$f''(x) = \frac{1}{h^2}(f(x+h) - 2f(x) + f(x-h)) - \frac{h^2}{12} f^{(4)}(\xi)$$

- Newton's method for solving the system of equations $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ is given by the scheme

$$\mathbf{J}^{(k)} \cdot \Delta\mathbf{x}^{(k)} = -\mathbf{f}(\mathbf{x}^{(k)})$$
$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \Delta\mathbf{x}^{(k)}$$

- Iteration methods for solving systems of linear equations

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \qquad i = 1, 2, \ldots, n$$

$$\text{Jacobi :} \quad x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right)$$

$$\text{Gauss-Seidel :} \quad x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right)$$

- Heun's method for solving $\mathbf{y}' = \mathbf{f}(x, \mathbf{y})$:

$$\mathbf{K}_1 = h\mathbf{f}(x_n, \mathbf{y}_n)$$
$$\mathbf{K}_2 = h\mathbf{f}(x_n + h, \mathbf{y}_n + \mathbf{K}_1)$$
$$\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{2}(\mathbf{K}_1 + \mathbf{K}_2)$$

See formulas in Rottmann as well.