# SIF5016 Matematikk 4N
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**Løsningsforslag**

Oppgavesettet har 11 punkter: 1, 2ab, 3ab, 4ab, 5, 6, 7ab som teller likt ved bedømmelsen.

**1** **a)** Finner ligningen for $f(t)$ fra grafen, og bruker formlene for Fourierkoeffisientene:

$$f(t)=\begin{cases}-\pi-t & \text{for } -\pi<t<0\\ t & \text{for } 0<t<\pi\end{cases}$$

$a_0=0$ (av grafen ser vi at $\int_{-\pi}^{\pi} f(t)\,dt=0$)

$$a_n=\frac{1}{\pi}\Big\{\int_{-\pi}^{0}-(t+\pi)\cos nt\,dt+\int_0^{\pi}t\cos nt\,dt\Big\}$$
$$=\frac{1}{\pi}\Big\{\Big[\underbrace{-(t+\pi)\frac{\sin nt}{n}}_{0}\Big]_{-\pi}^{0}+\int_{-\pi}^{0}\frac{\sin nt}{n}\,dt+\Big[\underbrace{t\frac{\sin nt}{n}}_{0}\Big]_0^{\pi}-\int_0^{\pi}\frac{\sin nt}{n}\,dt\Big\}$$
$$=\frac{1}{\pi}\Big\{\Big[-\frac{\cos nt}{n^2}\Big]_{-\pi}^{0}+\Big[\frac{\cos nt}{n^2}\Big]_0^{\pi}\Big\}=\frac{2}{\pi}\frac{(-1)^n-1}{n^2}=\begin{cases}0 & \text{for } n \text{ partall}\\ -4/(\pi n^2) & \text{for } n \text{ oddetall}\end{cases}$$

$$b_n=\frac{1}{\pi}\Big\{\int_{-\pi}^{0}-(t+\pi)\sin nt\,dt+\int_0^{\pi}t\sin nt\,dt\Big\}$$
$$=\frac{1}{\pi}\Big\{\Big[(t+\pi)\frac{\cos nt}{n}\Big]_{-\pi}^{0}-\underbrace{\int_{-\pi}^{0}\frac{\cos nt}{n}\,dt}_{0}-\Big[t\frac{\cos nt}{n}\Big]_0^{\pi}+\underbrace{\int_0^{\pi}\frac{\cos nt}{n}\,dt}_{0}\Big\}$$
$$=\frac{1}{\pi}\Big\{\pi\frac{1}{n}-\pi\frac{(-1)^n}{n}\Big\}=\frac{1-(-1)^n}{n}=\begin{cases}0 & \text{når } n \text{ er partall}\\ 2/n & \text{når } n \text{ er oddetall}\end{cases}$$

$$f(t)\sim\sum_{\substack{n=1\\ n\text{ odde}}}^{\infty}\Big(\frac{2}{n}\sin nt-\frac{4}{\pi n^2}\cos nt\Big)=\sum_{m=0}^{\infty}\Big(\frac{2\sin(2m+1)t}{2m+1}-\frac{4\cos(2m+1)t}{\pi(2m+1)^2}\Big)$$

**b)** La $S(t)$ være summen av rekka i **a)**.

$$S(0)=\tfrac{1}{2}\big[f(0+)+f(0-)\big]=\tfrac{1}{2}\big[0+(-\pi)\big]=-\frac{\pi}{2}\quad\text{(fra grafen i \textbf{a})}$$
$$S(\pi)=\tfrac{1}{2}\big[f(\pi+)+f(\pi-)\big]=\tfrac{1}{2}\big[0+\pi\big]=\frac{\pi}{2}$$
$$S(t)=f(t)=f(t-2\pi)=-\pi-(t-2\pi)=\pi-t,\quad \pi<t<2\pi$$

siden $f(t)$ er periodisk med periode $2\pi$ og $f(t)=-\pi-t$ for $-\pi<t<0$.





**2** **a)** Løs

$$y''+3y'+2y=1-u(t-1),\qquad y(0)=0,\quad y'(0)=0.$$

Bruker Laplacetransformasjonen:

$$\mathcal{L}\{y\}=Y,\qquad s^2Y+3sY+2Y=\frac{1}{s}-\frac{1}{s}e^{-s}$$
$$Y=\frac{1}{s^2+3s+2}\cdot\frac{1}{s}(1-e^{-s})=\frac{1}{(s+1)(s+2)s}(1-e^{-s})=\Big[\frac{1/2}{s+2}-\frac{1}{s+1}+\frac{1/2}{s}\Big](1-e^{-s})$$
$$\mathcal{L}^{-1}\Big\{\frac{1}{s+1}\Big\}=e^{-t},\qquad \mathcal{L}^{-1}\Big\{\frac{1}{s+2}\Big\}=e^{-2t},\qquad \mathcal{L}^{-1}\Big\{\frac{1}{s}\Big\}=1$$
$$y=\mathcal{L}^{-1}\{Y\}=\tfrac{1}{2}e^{-2t}-e^{-t}+\tfrac{1}{2}-\Big[\tfrac{1}{2}e^{-2(t-1)}-e^{-(t-1)}+\tfrac{1}{2}\Big]u(t-1)$$
$$y=\begin{cases}\frac{1}{2}e^{-2t}-e^{-t}+\frac{1}{2} & \text{for } t<1\\ \frac{1}{2}(1-e^2)e^{-2t}-(1-e)e^{-t} & \text{for } t\ge 1\end{cases}$$

**b)** Laplacetransformerer ligningen ved å bruke konvolusjonsregelen:

$$y(t)=\cos t+y(t)*\cos t,\quad \mathcal{L}\{y\}=Y,\quad \mathcal{L}\{\cos t\}=\frac{s}{s^2+1}$$
$$Y=\frac{s}{s^2+1}+Y\cdot\frac{s}{s^2+1},\quad Y\Big(1-\frac{s}{s^2+1}\Big)=\frac{s}{s^2+1},\quad Y\Big(\frac{s^2-s+1}{s^2+1}\Big)=\frac{s}{s^2+1}$$
$$Y=\frac{s}{s^2-s+1}=\frac{s}{(s-1/2)^2+3/4}=\frac{(s-1/2)+1/2}{(s-1/2)^2+(\sqrt{3}/2)^2}$$
$$y=\mathcal{L}^{-1}\{Y\}=e^{\frac{t}{2}}\left(\cos\frac{\sqrt{3}}{2}t+\frac{1}{2}\cdot\frac{1}{\sqrt{3}/2}\sin\frac{\sqrt{3}}{2}t\right)=e^{\frac{t}{2}}\left(\cos\frac{\sqrt{3}}{2}t+\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right)$$

**3** **a)** Separasjon av variable:

$$u(x,t)=F(x)\cdot G(t):\qquad FG''-F''G+FG=0,\quad F'(0)=0,\ F'(\pi)=0$$
$$\frac{G''}{G}+1=\frac{F''}{F}=-\lambda\quad\text{(konstant)}$$
(I) $F''+\lambda F=0,\quad F'(0)=0,\ F'(\pi)=0,\qquad$ (II) $G''+(\lambda+1)G=0$

(I) $\lambda<0,\ \lambda=-\alpha^2:\quad F(x)=Ae^{\alpha x}+Be^{-\alpha x},\ F'(x)=\alpha Ae^{\alpha x}-\alpha Be^{-\alpha x}$
$F'(0)=0,\ F'(\pi)=0\Rightarrow A=B=0,\ F(x)=0$
$\lambda=0:\quad F(x)=Ax+B,\ F'(x)=A,\quad F'(0)=F'(\pi)=0\Rightarrow A=0,\ F_0(x)=1$
$\lambda>0,\ \lambda=\alpha^2:\quad F(x)=A\cos\alpha x+B\sin\alpha x,\ F'(x)=-\alpha A\sin\alpha x+\alpha B\cos\alpha x$
$F'(0)=0\Rightarrow B=0,\ F'(\pi)=0\Rightarrow\alpha=n,\ \lambda=n^2,\ F_n(x)=\cos nx,\ n\ge 1$

(II) $\lambda=0:\quad G''+G=0,\ G_0(t)=C_0\cos t+D_0\sin t$
$\lambda=n^2:\quad G''+(n^2+1)G=0,\ G_n(t)=C_n\cos\sqrt{n^2+1}\,t+D_n\sin\sqrt{n^2+1}\,t$

$$u_0(x,t) = C_0\cos t + D_0\sin t$$
$$u_n(x,t) = \left(C_n\cos\sqrt{n^2+1}\,t + D_n\sin\sqrt{n^2+1}\,t\right)\cos nx,\ n \geq 1$$
dvs. $u_n(x,t) = \left(C_n\cos\sqrt{n^2+1}\,t + D_n\sin\sqrt{n^2+1}\,t\right)\cos nx,\ n = 0,1,2,\ldots$

**b)** Superposisjonprinsippet:

$$u(x,t) = \sum_{n=0}^{\infty}\left(C_n\cos\sqrt{n^2+1}\,t + D_n\sin\sqrt{n^2+1}\,t\right)\cos nx$$
$$u_t(x,t) = \sum_{n=0}^{\infty}\left(-C_n\sqrt{n^2+1}\sin\sqrt{n^2+1}\,t + D_n\sqrt{n^2+1}\cos\sqrt{n^2+1}\,t\right)\cos nx$$
$$1+2\cos x = u(x,0) = \sum_{n=0}^{\infty} C_n\cos nx \Rightarrow C_0 = 1,\ C_1 = 2,\ C_n = 0 \text{ for } n \geq 2$$
$$\cos 2x = u_t(x,0) = \sum_{n=0}^{\infty} D_n\sqrt{n^2+1}\cos nx \Rightarrow D_2 = 1/\sqrt{5},\ D_n = 0 \text{ for } n \neq 2$$
$$u(x,t) = \cos t + 2\cos\sqrt{2}t\cos x + \frac{1}{\sqrt{5}}\sin\sqrt{5}t\cos 2x$$

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$$\begin{aligned} p_3(x) &= 4\tfrac{x(x-1)(x-2)}{(-1)\cdot(-2)\cdot(-3)} + 1\tfrac{(x+1)(x-1)(x-2)}{1\cdot(-1)\cdot(-2)} \\ &\quad -2\tfrac{(x+1)x(x-2)}{2\cdot1\cdot(-1)} + 1\tfrac{(x+1)x(x-1)}{3\cdot2\cdot1} \\ &= -\tfrac{2}{3}(x^3-3x^2+2x) + \tfrac{1}{2}(x^3-2x^2-x+2) \\ &\quad +(x^3-x^2-2x) + \tfrac{1}{6}(x^3-x) \\ &= x^3-4x+1 \end{aligned} \tag{1}$$

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$$J = \begin{bmatrix} 2x+y^3 & 3y^2x \\ 6xy & 3x^2-3y^2 \end{bmatrix} \tag{2}$$

For $k = 0,1,2$ løs

$$\begin{bmatrix} 2x^{(k)}+(y^{(k)})^3 & 3(y^{(k)})^2x^{(k)} \\ 6x^{(k)}y^{(k)} & 3(x^{(k)})^2-3(y^{(k)})^2 \end{bmatrix}\begin{bmatrix}\Delta x^{(k)}\\ \Delta y^{(k)}\end{bmatrix} = -\begin{bmatrix}(x^{(k)})^2+x^{(k)}(y^{(k)})^3-9\\ 3(x^{(k)})^2y^{(k)}-(y^{(k)})^3-4\end{bmatrix}, \tag{3}$$

$$\begin{aligned} x^{(k+1)} &= x^{(k)}+\Delta x^{(k)} \\ y^{(k+1)} &= y^{(k)}+\Delta y^{(k)} \end{aligned} \tag{4}$$

- Først iterasjon

$$\begin{bmatrix} 18.025 & 22.5 \\ 18.00 & -14.43 \end{bmatrix}\begin{bmatrix}\Delta x^{(0)}\\ \Delta y^{(0)}\end{bmatrix} = \begin{bmatrix}-11.19\\ 8.825\end{bmatrix}, \tag{5}$$

og $\Delta x^{(0)} = 0.05577,\ x^{(1)} = 1.25577,\ \Delta y^{(0)} = -0.542009,\ y^{(1)} = 1.95799.$





- Andre iterasjon

$$\begin{bmatrix} 10.018 & 14.444 \\ 14.753 & -6.7700 \end{bmatrix}\begin{bmatrix}\Delta x^{(1)}\\ \Delta y^{(1)}\end{bmatrix} = \begin{bmatrix}-2.004\\ 2.2431\end{bmatrix}, \tag{6}$$

og $\Delta x^{(1)} = 0.067,\ x^{(2)} = 1.3228,\ \Delta y^{(1)} = -0.185,\ y^{(2)} = 1.7728.$

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**a)** Vi vil løse

$$u_t = u_{xx},\ \text{for } 0 \leq x \leq 1 \text{ og } t \geq 0,$$

med $u(x,0) = \sin(\pi x)$, $0 \leq x \leq 1$ og $u(0,t) = u(1,t) = 0$, $t \geq 0$.
Vi bruke Crank-Nicolson metode:

$$\frac{1}{k}(u(x,t)-u(x,t-k)) = \frac{1}{h^2}(u(x+h,t)-2u(x,t)+u(x-h,t)). \tag{7}$$

Med $k = h = 1/4$ og gridpunkter $x_i = i\cdot h$, $t_j = j\cdot k$, la $u_i^j \approx u(x_i,t_j)$, og fra

$$\frac{1}{k}(u_i^j-u_i^{j-1}) = \frac{1}{h^2}(u_{i+1}^j-2u_i^j+u_{i-1}^j).$$

Hvis vi antar at $u_i^{j-1}$ er kjent, ($s = \frac{h^2}{k}$), får vi

$$-u_{i+1}^j+(2+s)u_i^j-u_{i-1}^j = su_i^{j-1},\ i = 1,\ldots,n-1,\ u_0^j = u_n^j = 0.$$

For $k = h = 0.25$, $s = 0.25$, $n = 4$

$$\begin{bmatrix} 2.25 & -1 & 0 \\ -1 & 2.25 & -1 \\ 0 & -1 & 2.25 \end{bmatrix}\begin{bmatrix}u_1^1\\ u_2^1\\ u_3^1\end{bmatrix} = 0.25\begin{bmatrix}u_1^0\\ u_2^0\\ u_3^0\end{bmatrix} = \begin{bmatrix}0.17678\\ 0.25\\ 0.17678\end{bmatrix}. \tag{8}$$

Men $u_1^0 = \sin(\pi\cdot0.25)$, $u_2^0 = \sin(\pi\cdot0.5)$, $u_3^0 = \sin(\pi\cdot0.75)$.

**b)**

```
A = [2.25, -1, 0; -1, 2.25, -1; 0, -1, 2.25];
b = 0.25*[sin(0.25*pi); sin(0.5*pi); sin(0.75*pi)];
u = A\b;
```