SIF5016 Matematikk 4N
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Norges teknisk-naturvitenskapelige universitet
Institutt for matematiske fag

Løsningsforslag

Oppgavesettet har 11 punkter: 1, 2ab, 3ab, 4ab, 5, 6, 7ab som teller likt ved bedømmelsen.

1 **a)** Finner ligningen for $f(t)$ fra grafen, og bruker formlene for Fourierkoeffisientene:

$$f(t) = \begin{cases} -\pi - t & \text{for } -\pi < t < 0 \\ t & \text{for } 0 < t < \pi \end{cases}$$

$$a_0 = 0 \quad \text{(av grafen ser vi at } \int_{-\pi}^{\pi} f(t)\,dt = 0\text{)}$$

$$a_n = \frac{1}{\pi}\left\{ \int_{-\pi}^{0} -(t+\pi)\cos nt\,dt + \int_0^{\pi} t\cos nt\,dt \right\}$$

$$= \frac{1}{\pi}\left\{ \underbrace{\left[-(t+\pi)\frac{\sin nt}{n} \right]_{-\pi}^{0}}_{0} + \int_{-\pi}^{0} \frac{\sin nt}{n}\,dt + \underbrace{\left[t\frac{\sin nt}{n} \right]_0^{\pi}}_{0} - \int_0^{\pi} \frac{\sin nt}{n}\,dt \right\}$$

$$= \frac{1}{\pi}\left\{ \left[-\frac{\cos nt}{n^2} \right]_{-\pi}^{0} + \left[\frac{\cos nt}{n^2} \right]_0^{\pi} \right\} = \frac{2}{\pi}\frac{(-1)^n - 1}{n^2} = \begin{cases} 0 & \text{for } n \text{ partall} \\ -4/(\pi n^2) & \text{for } n \text{ oddetall} \end{cases}$$

$$b_n = \frac{1}{\pi}\left\{ \int_{-\pi}^{0} -(t+\pi)\sin nt\,dt + \int_0^{\pi} t\sin nt\,dt \right\}$$

$$= \frac{1}{\pi}\left\{ \left[(t+\pi)\frac{\cos nt}{n} \right]_{-\pi}^{0} - \underbrace{\int_{-\pi}^{0} \frac{\cos nt}{n}\,dt}_{0} - \left[t\frac{\cos nt}{n} \right]_0^{\pi} + \underbrace{\int_0^{\pi} \frac{\cos nt}{n}\,dt}_{0} \right\}$$

$$= \frac{1}{\pi}\left\{ \pi\frac{1}{n} - \pi\frac{(-1)^n}{n} \right\} = \frac{1-(-1)^n}{n} = \begin{cases} 0 & \text{når } n \text{ er partall} \\ 2/n & \text{når } n \text{ er oddetall} \end{cases}$$

$$f(t) \sim \sum_{\substack{n=1 \\ n \text{ odde}}}^{\infty} \left(\frac{2}{n}\sin nt - \frac{4}{\pi n^2}\cos nt \right) = \sum_{m=0}^{\infty} \left(\frac{2\sin(2m+1)t}{2m+1} - \frac{4\cos(2m+1)t}{\pi(2m+1)^2} \right)$$

b) La $S(t)$ være summen av rekka i **a)**.

$$S(0) = \tfrac{1}{2}\left[f(0+) + f(0-) \right] = \tfrac{1}{2}\left[0 + (-\pi) \right] = -\frac{\pi}{2} \quad \text{(fra grafen i \textbf{a)})}$$

$$S(\pi) = \tfrac{1}{2}\left[f(\pi+) + f(\pi-) \right] = \tfrac{1}{2}\left[0 + \pi \right] = \frac{\pi}{2}$$

$$S(t) = f(t) = f(t-2\pi) = -\pi - (t-2\pi) = \pi - t, \quad \pi < t < 2\pi$$

siden $f(t)$ er periodisk med periode 2π og $f(t) = -\pi - t$ for $-\pi < t < 0$.

2 **a)** Løs

$$y'' + 3y' + 2y = 1 - u(t-1), \qquad y(0) = 0, \quad y'(0) = 0.$$

Bruker Laplacetransformasjonen:

$$L\{y\} = Y, \qquad s^2Y + 3sY + 2Y = \frac{1}{s} - \frac{1}{s}e^{-s}$$

$$Y = \frac{1}{s^2+3s+2} \cdot \frac{1}{s}(1-e^{-s}) = \frac{1}{(s+1)(s+2)s}(1-e^{-s}) = \Big[\frac{1/2}{s+2} - \frac{1}{s+1} + \frac{1/2}{s}\Big](1-e^{-s})$$

$$L^{-1}\Big\{\frac{1}{s+1}\Big\} = e^{-t}, \qquad L^{-1}\Big\{\frac{1}{s+2}\Big\} = e^{-2t}, \qquad L^{-1}\Big\{\frac{1}{s}\Big\} = 1$$

$$y = L^{-1}\{Y\} = \tfrac{1}{2}e^{-2t} - e^{-t} + \tfrac{1}{2} - \Big[\tfrac{1}{2}e^{-2(t-1)} - e^{-(t-1)} + \tfrac{1}{2}\Big]u(t-1)$$

$$y = \begin{cases} \frac{1}{2}e^{-2t} - e^{-t} + \frac{1}{2} & \text{for } t < 1 \\ \frac{1}{2}(1-e^2)e^{-2t} - (1-e)e^{-t} & \text{for } t \geq 1 \end{cases}$$

b) Laplacetransformerer ligningen ved å bruke konvolusjonsregelen:

$$y(t) = \cos t + y(t) * \cos t, \quad L\{y\} = Y, \quad L\{\cos t\} = \frac{s}{s^2+1}$$

$$Y = \frac{s}{s^2+1} + Y \cdot \frac{s}{s^2+1}, \quad Y\Big(1 - \frac{s}{s^2+1}\Big) = \frac{s}{s^2+1}, \quad Y\Big(\frac{s^2-s+1}{s^2+1}\Big) = \frac{s}{s^2+1}$$

$$Y = \frac{s}{s^2-s+1} = \frac{s}{(s-1/2)^2+3/4} = \frac{(s-1/2)+1/2}{(s-1/2)^2+(\sqrt{3}/2)^2}$$

$$y = L^{-1}\{Y\} = e^{\frac{t}{2}}\left(\cos\frac{\sqrt{3}}{2}t + \frac{1}{2}\cdot\frac{1}{\sqrt{3}/2}\sin\frac{\sqrt{3}}{2}t\right) = e^{\frac{t}{2}}\left(\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right)$$

3 **a)** Separasjon av variable:

$$u(x,t) = F(x)\cdot G(t): \qquad FG'' - F''G + FG = 0, \quad F'(0) = 0,\ F'(\pi) = 0$$

$$\frac{G''}{G} + 1 = \frac{F''}{F} = -\lambda \quad \text{(konstant)}$$

$$\text{(I)} \quad F'' + \lambda F = 0, \quad F'(0) = 0,\ F'(\pi) = 0, \qquad \text{(II)} \quad G'' + (\lambda+1)G = 0$$

(I) $\lambda < 0,\ \lambda = -\alpha^2: \quad F(x) = Ae^{\alpha x} + Be^{-\alpha x},\ F'(x) = \alpha Ae^{\alpha x} - \alpha Be^{-\alpha x}$

$F'(0) = 0,\ F'(\pi) = 0 \Rightarrow A = B = 0,\ F(x) = 0$

$\lambda = 0: \quad F(x) = Ax + B,\ F'(x) = A, \quad F'(0) = F'(\pi) = 0 \Rightarrow A = 0,\ F_0(x) = 1$

$\lambda > 0,\ \lambda = \alpha^2: \quad F(x) = A\cos\alpha x + B\sin\alpha x,\ F'(x) = -\alpha A\sin\alpha x + \alpha B\cos\alpha x$

$F'(0) = 0 \Rightarrow B = 0,\ F'(\pi) = 0 \Rightarrow \alpha = n,\ \lambda = n^2,\ F_n(x) = \cos nx,\ n \geq 1$

(II) $\lambda = 0: \quad G'' + G = 0,\ G_0(t) = C_0\cos t + D_0\sin t$

$\lambda = n^2: \quad G'' + (n^2+1)G = 0,\ G_n(t) = C_n\cos\sqrt{n^2+1}\,t + D_n\sin\sqrt{n^2+1}\,t$

$$u_0(x,t) = C_0 \cos t + D_0 \sin t$$
$$u_n(x,t) = \left(C_n \cos \sqrt{n^2+1}\,t + D_n \sin \sqrt{n^2+1}\,t\right) \cos nx,\ n \geq 1$$
$$\text{dvs. } u_n(x,t) = \left(C_n \cos \sqrt{n^2+1}\,t + D_n \sin \sqrt{n^2+1}\,t\right) \cos nx,\ n = 0,1,2,\ldots$$

b) Superposisjonprinsippet:

$$u(x,t) = \sum_{n=0}^{\infty} \left(C_n \cos \sqrt{n^2+1}\,t + D_n \sin \sqrt{n^2+1}\,t\right) \cos nx$$
$$u_t(x,t) = \sum_{n=0}^{\infty} \left(-C_n\sqrt{n^2+1} \sin \sqrt{n^2+1}\,t + D_n\sqrt{n^2+1} \cos \sqrt{n^2+1}\,t\right) \cos nx$$
$$1 + 2\cos x = u(x,0) = \sum_{n=0}^{\infty} C_n \cos nx \Rightarrow C_0 = 1,\ C_1 = 2,\ C_n = 0 \text{ for } n \geq 2$$
$$\cos 2x = u_t(x,0) = \sum_{n=0}^{\infty} D_n\sqrt{n^2+1} \cos nx \Rightarrow D_2 = 1/\sqrt{5},\ D_n = 0 \text{ for } n \neq 2$$
$$u(x,t) = \cos t + 2\cos\sqrt{2}\,t \cos x + \frac{1}{\sqrt{5}} \sin\sqrt{5}\,t \cos 2x$$

5

$$\begin{aligned} p_3(x) &= 4\tfrac{x(x-1)(x-2)}{(-1)\cdot(-2)\cdot(-3)} + 1\tfrac{(x+1)(x-1)(x-2)}{1\cdot(-1)\cdot(-2)} \\ &\quad -2\tfrac{(x+1)x(x-2)}{2\cdot1\cdot(-1)} + 1\tfrac{(x+1)x(x-1)}{3\cdot2\cdot1} \\ &= -\tfrac{2}{3}(x^3-3x^2+2x) + \tfrac{1}{2}(x^3-2x^2-x+2) \\ &\quad +(x^3-x^2-2x) + \tfrac{1}{6}(x^3-x) \\ &= x^3-4x+1 \end{aligned} \tag{1}$$

6

$$J = \begin{bmatrix} 2x+y^3 & 3y^2x \\ 6xy & 3x^2-3y^2 \end{bmatrix} \tag{2}$$

For $k = 0,1,2$ løs

$$\begin{bmatrix} 2x^{(k)}+(y^{(k)})^3 & 3(y^{(k)})^2x^{(k)} \\ 6x^{(k)}y^{(k)} & 3(x^{(k)})^2-3(y^{(k)})^2 \end{bmatrix} \begin{bmatrix} \Delta x^{(k)} \\ \Delta y^{(k)} \end{bmatrix} = -\begin{bmatrix} (x^{(k)})^2+x^{(k)}(y^{(k)})^3-9 \\ 3(x^{(k)})^2y^{(k)}-(y^{(k)})^3-4 \end{bmatrix}, \tag{3}$$

$$\begin{aligned} x^{(k+1)} &= x^{(k)}+\Delta x^{(k)} \\ y^{(k+1)} &= y^{(k)}+\Delta y^{(k)} \end{aligned} \tag{4}$$

- Først iterasjon

$$\begin{bmatrix} 18.025 & 22.5 \\ 18.00 & -14.43 \end{bmatrix} \begin{bmatrix} \Delta x^{(0)} \\ \Delta y^{(0)} \end{bmatrix} = \begin{bmatrix} -11.19 \\ 8.825 \end{bmatrix}, \tag{5}$$

og $\Delta x^{(0)} = 0.05577,\ x^{(1)} = 1.25577,\ \Delta y^{(0)} = -0.542009,\ y^{(1)} = 1.95799.$

- Andre iterasjon

$$\begin{bmatrix} 10.018 & 14.444 \\ 14.753 & -6.7700 \end{bmatrix} \begin{bmatrix} \Delta x^{(1)} \\ \Delta y^{(1)} \end{bmatrix} = \begin{bmatrix} -2.004 \\ 2.2431 \end{bmatrix}, \tag{6}$$

og $\Delta x^{(1)} = 0.067,\ x^{(2)} = 1.3228$, $\Delta y^{(1)} = -0.185,\ y^{(2)} = 1.7728$.

7 **a)** Vi vil løse

$$u_t = u_{xx},\ \text{for } 0 \leq x \leq 1 \text{ og } t \geq 0,$$

med $u(x,0) = \sin(\pi x)$, $0 \leq x \leq 1$ og $u(0,t) = u(1,t) = 0$, $t \geq 0$.
Vi bruke Crank-Nicolson metode:

$$\frac{1}{k}(u(x,t) - u(x,t-k)) = \frac{1}{h^2}(u(x+h,t) - 2u(x,t) + u(x-h,t)). \tag{7}$$

Med $k = h = 1/4$ og gridpunkter $x_i = i \cdot h$, $t_j = j \cdot k$, la $u_i^j \approx u(x_i, t_j)$, og fra

$$\frac{1}{k}(u_i^j - u_i^{j-1}) = \frac{1}{h^2}(u_{i+1}^j - 2u_i^j + u_{i-1}^j).$$

Hvis vi antar at u_i^{j-1} er kjent, ($s = \frac{h^2}{k}$), får vi

$$-u_{i+1}^j + (2+s)u_i^j - u_{i-1}^j = su_i^{j-1},\ i = 1, \ldots, n-1,\ u_0^j = u_n^j = 0.$$

For $k = h = 0.25$, $s = 0.25$, $n = 4$

$$\begin{bmatrix} 2.25 & -1 & 0 \\ -1 & 2.25 & -1 \\ 0 & -1 & 2.25 \end{bmatrix} \begin{bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \end{bmatrix} = 0.25 \begin{bmatrix} u_1^0 \\ u_2^0 \\ u_3^0 \end{bmatrix} = \begin{bmatrix} 0.17678 \\ 0.25 \\ 0.17678 \end{bmatrix}. \tag{8}$$

Men $u_1^0 = \sin(\pi \cdot 0.25)$, $u_2^0 = \sin(\pi \cdot 0.5)$, $u_3^0 = \sin(\pi \cdot 0.75)$.

b)

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A = [2.25, -1, 0; -1, 2.25, -1; 0, -1, 2.25];
b = 0.25*[sin(0.25*pi); sin(0.5*pi); sin(0.75*pi)];
u = A\b;
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