Fra Kreyszig, avsnitt 10.7

7 Funksjonen $f(x)$ er odde så Fourierrekka blir ei sinusrekke.

Ved å bruke formel 121 i Rottmann s. 144 (eller delvis integrasjon) finn vi

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi/2} x\sin nx\,dx \\ &= \frac{2}{\pi}\left[\frac{1}{n^2}\sin nx - \frac{x}{n}\cos nx\right]_0^{\pi/2} = \frac{2}{\pi}\left[\frac{1}{n^2}\sin\frac{n\pi}{2} - \frac{\pi}{2n}\cos\frac{n\pi}{2}\right]. \end{aligned}$$

Dette gir

$$\begin{aligned} b_{2m-1} &= \frac{2}{\pi}\frac{\sin(m-\frac{1}{2})\pi}{(2m-1)^2} = \frac{2(-1)^{m-1}}{\pi(2m-1)^2} \qquad (m = 1, 2, 3, \dots) \\ b_{2m} &= \frac{2}{\pi}\left[-\frac{\pi\cos m\pi}{4m}\right] = \frac{(-1)^{m+1}}{2m} \qquad (m = 1, 2, 3, \dots). \end{aligned}$$

Rekkeutviklinga blir derfor (merk at $(-1)^{m-1} = (-1)^{m+1}$):

$$\begin{aligned} f(x) &\sim \sum_{m=1}^{\infty}(-1)^{m+1}\left[\frac{2}{\pi(2m-1)^2}\sin(2m-1)x + \frac{1}{2m}\sin 2mx\right] \\ &= \frac{2}{\pi}\sin x + \frac{1}{2}\sin 2x - \frac{2}{9\pi}\sin 3x - \frac{1}{4}\sin 4x + \frac{2}{25\pi}\sin 5x + \cdots. \end{aligned}$$

Vi har altså

$$b_1 = \frac{2}{\pi}, \quad b_2 = \frac{1}{2}, \quad b_3 = -\frac{2}{9\pi}, \quad b_4 = -\frac{1}{4}, \quad b_5 = \frac{2}{25\pi}.$$

Kvadratfeilen får sitt minimum E_N^* for Fourierpolynomet $F_N(x) = \sum_{n=1}^{N} b_n \sin nx$. Altså

$$\begin{aligned} F_1(x) &= \frac{2}{\pi}\sin x \\ F_2(x) &= \frac{2}{\pi}\sin x + \frac{1}{2}\sin 2x \\ F_3(x) &= \frac{2}{\pi}\sin x + \frac{1}{2}\sin 2x - \frac{2}{9\pi}\sin 3x \\ F_4(x) &= \frac{2}{\pi}\sin x + \frac{1}{2}\sin 2x - \frac{2}{9\pi}\sin 3x - \frac{1}{4}\sin 4x \\ F_5(x) &= \frac{2}{\pi}\sin x + \frac{1}{2}\sin 2x - \frac{2}{9\pi}\sin 3x - \frac{1}{4}\sin 4x + \frac{2}{25\pi}\sin 5x. \end{aligned}$$

Etter formel (6) i Kreyszig 10.7 har vi

$$E_N^* = \int_{-\pi}^{\pi} f(x)^2\,dx - \pi\sum_{n=1}^{N} b_n^2.$$

Her er $\int_{-\pi}^{\pi} f(x)^2\,dx = \int_{-\pi/2}^{\pi/2} x^2\,dx = \pi^3/12$, og vi legg merke til at $E^*_{N+1} = E^*_N - \pi b^2_{N+1}$. Vi får

$$E_1^* = \pi^3/12 - \pi b_1^2 \approx 1.311, \qquad E_2^* = E_1^* - \pi b_2^2 \approx 0.525, \qquad E_3^* = E_2^* - \pi b_3^2 \approx 0.509$$
$$E_4^* = E_3^* - \pi b_4^2 \approx 0.313, \qquad E_5^* = E_4^* - \pi b_5^2 \approx 0.311.$$

11 Fra Kreyszig 10.2 oppgave 7 med fasit har vi

$$\begin{aligned} x^2 &= \frac{\pi^2}{3} - 4\Big(\cos x - \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x - \frac{1}{16}\cos 4x + - \cdots\Big) \\ &= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx, \qquad -\pi \le x \le \pi. \end{aligned}$$

Parsevals identitet, Kreyszig 10.7 formel (8), gir da

$$\begin{aligned} 2\left(\frac{\pi^2}{3}\right)^2 + \sum_{n=1}^{\infty}\left(4\frac{(-1)^n}{n^2}\right)^2 &= \frac{1}{\pi}\int_{-\pi}^{\pi} \left(x^2\right)^2\,dx, \\ \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{1}{16}\left(2\cdot\frac{\pi^4}{5} - 2\cdot\frac{\pi^4}{9}\right) &= \frac{\pi^4}{90} \qquad (\approx 1.082). \end{aligned}$$

De fire første partialsummene er

$$S_1 = 1, \quad S_2 = S_1 + 1/2^4 \approx 1.063, \quad S_3 = S_2 + 1/3^4 \approx 1.075, \quad S_4 = S_3 + 1/4^4 \approx 1.079.$$

Fra Kreyszig, avsnitt 10.8

17 Vi bruker formel (12) i på side 562 i Kreyszig, samt relasjonene $\cos x = (e^{ix} + e^{-ix})/2$ og $\sin x = (e^{ix} - e^{-ix})/2i$:

$$\begin{aligned} B(w) &= \frac{2}{\pi}\int_0^{\infty} f(v)\sin wv dv = \frac{2}{\pi}\int_0^1 e^v \sin wv dv \\ &= \frac{2}{\pi}\frac{1}{2i}\int_0^1 \left[e^{v(1+iw)} - e^{v(1-iw)}\right] dv \\ &= \frac{2}{\pi}\frac{1}{2i}\left[\frac{1}{1+iw}e^{v(1+iw)} - \frac{1}{1-iw}e^{v(1-iw)}\right]_{v=0}^{1} \\ &= \frac{2}{\pi}\frac{1}{2i}\left(\frac{1}{1+iw}e^{1+iw} - \frac{1}{1-iw}e^{1-iw} - \frac{1}{1+iw} + \frac{1}{1-iw}\right) \\ &= \frac{2}{\pi}\frac{1}{2i}\left(e\frac{(1+iw)e^{iw} - (1-iw)e^{-iw}}{1+w^2} + \frac{2iw}{1+w^2}\right) \\ &= \frac{2}{\pi}\frac{1}{1+w^2}\frac{1}{2i}\left[e\underbrace{(e^{iw} - e^{-iw})}_{=2i\sin w} - ew\underbrace{i(e^{iw} + e^{-iw})}_{=2i\cos w} + 2iw\right] \\ &= \frac{2}{\pi}\frac{w - e(w\cos w - \sin w)}{1+w^2}, \end{aligned}$$

så, på formen (13) på side 562 i Kreyszig:

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \frac{w - e(w\cos w - \sin w)}{1+w^2}\sin xw dw$$

Eksamensoppgaver

4 **a)** Finn først likninga for $f(t)$ frå grafen.

$$f(t) = \begin{cases} -\pi - t & \text{for } -\pi < t < 0 \\ t & \text{for } 0 < t < \pi \end{cases}$$

Ved hjelp av (6) side 531 i Kreyszig finn me Fourierkoeffisientane.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt = 0 \quad \text{(ser direkte frå grafen)}$$

$$a_n = \frac{1}{\pi}\left\{\int_{-\pi}^{0} -(t+\pi)\cos nt\,dt + \int_{0}^{\pi} t\cos nt\,dt\right\}$$

$$= \frac{1}{\pi}\left\{\underbrace{\left[-(t+\pi)\frac{\sin nt}{n}\right]_{-\pi}^{0}}_{0} + \int_{-\pi}^{0}\frac{\sin nt}{n}\,dt + \underbrace{\left[t\frac{\sin nt}{n}\right]_{0}^{\pi}}_{0} - \int_{0}^{\pi}\frac{\sin nt}{n}\,dt\right\}$$

$$= \frac{1}{\pi}\left\{\left[-\frac{\cos nt}{n^2}\right]_{-\pi}^{0} + \left[\frac{\cos nt}{n^2}\right]_{0}^{\pi}\right\} = \frac{2}{\pi}\frac{(-1)^n - 1}{n^2} = \begin{cases} 0 & \text{for } n \text{ jamn} \\ -\frac{4}{\pi n^2} & \text{for } n \text{ odde} \end{cases}$$

$$b_n = \frac{1}{\pi}\left\{\int_{-\pi}^{0} -(t+\pi)\sin nt\,dt + \int_{0}^{\pi} t\sin nt\,dt\right\}$$

$$= \frac{1}{\pi}\left\{\left[(t+\pi)\frac{\cos nt}{n}\right]_{-\pi}^{0} - \underbrace{\int_{-\pi}^{0}\frac{\cos nt}{n}\,dt}_{0} - \left[t\frac{\cos nt}{n}\right]_{0}^{\pi} + \underbrace{\int_{0}^{\pi}\frac{\cos nt}{n}\,dt}_{0}\right]$$

$$= \frac{1}{\pi}\left\{\pi\frac{1}{n} - \pi\frac{(-1)^n}{n}\right\} = \frac{1-(-1)^n}{n} = \begin{cases} 0 & \text{for } n \text{ jamn} \\ \frac{2}{n} & \text{for } n \text{ odde} \end{cases}$$

$$f(t) \sim \sum_{\substack{n=1 \\ n \text{ odde}}}^{\infty}\left(\frac{2}{n}\sin nt - \frac{4}{\pi n^2}\cos\ nt\right) = \sum_{m=0}^{\infty}\left(\frac{2\sin(2m+1)t}{2m+1} - \frac{4\cos(2m+1)t}{\pi(2m+1)^2}\right)$$

b) La $S(t)$ vera summen av rekka i **a)**.

$$S(0) = \frac{1}{2}\left[f(0+) + f(0-)\right] = \frac{1}{2}\left[0 + (-\pi)\right] = -\frac{\pi}{2} \quad \text{(frå grafen i **a)**)}.$$

$$S(\pi) = \frac{1}{2}\left[f(\pi+) + f(\pi-)\right] = \frac{1}{2}\left[0 + \pi\right] = \frac{\pi}{2}$$

$$S(t) = f(t) = f(t-2\pi) = -\pi - (t-2\pi) = \pi - t, \quad \pi < t < 2\pi$$

sidan $f(t)$ er periodisk med periode 2π og $f(t) = -\pi - t$ for $-\pi < t < 0$.

5 Fra den komplekse Fourierrekka kan vi finne den vanlige Fourierrekka,

$$f(x) \sim \frac{\sinh\pi}{\pi}\Big[\sum_{n=-1}^{-\infty}(-1)^n\frac{1+in}{1+n^2}e^{inx} + 1 + \sum_{n=1}^{\infty}(-1)^n\frac{1+in}{1+n^2}e^{inx}\Big]$$

$$= \frac{\sinh\pi}{\pi}\Big[\sum_{n=1}^{\infty}(-1)^n\frac{1-in}{1+(-n)^2}e^{-inx} + 1 + \sum_{n=1}^{\infty}(-1)^n\frac{1+in}{1+n^2}e^{inx}\Big]$$

$$= \frac{\sinh\pi}{\pi}\Big[1 + \sum_{n=1}^{\infty}(-1)^n\frac{(1-in)e^{-inx} + (1+in)e^{inx}}{1+n^2}\Big]$$

$$= \frac{2\sinh\pi}{\pi}\Big[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos nx - n\sin nx)\Big]$$

Setter inn $x = 0$:

$$1 = \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(1-0)\right] \quad \text{som gir} \quad \sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2} = \frac{\pi}{2\sinh\pi} - \frac{1}{2}$$

Setter inn $x = \pi$ og bruker at $\lim_{x\to\pi-} f(x) = e^{\pi}, \lim_{x\to\pi+} f(x) = e^{-\pi}$:

$$\frac{e^{\pi}+e^{-\pi}}{2} = \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\cdot(-1)^n\right] \quad \text{som gir} \quad \sum_{n=1}^{\infty}\frac{1}{1+n^2} = \frac{\pi\cosh\pi}{2\sinh\pi} - \frac{1}{2}$$