**Løsningsforslag - Øving 6**

## Eksamensoppgaver

**1** **a)** Den Fouriertransformerte av $f(x)$ er

$$\widehat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{e}^{-(1+\mathrm{i}w)x}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \left[ -\frac{\mathrm{e}^{-(1+iw)x}}{1+\mathrm{i}w} \right]_0^\infty = \frac{1}{\sqrt{2\pi}\,(1+\mathrm{i}w)}.$$

(Vi brukte at $\lim_{x\to\infty} \mathrm{e}^{-(1+\mathrm{i}w)x} = \lim_{x\to\infty} \mathrm{e}^{-x}\mathrm{e}^{-\mathrm{i}wx} = \lim_{x\to\infty} \mathrm{e}^{-x}(\cos wx - \sin wx) = 0$.)
Siden $\widehat{h}(w) = 2\pi\,\widehat{f}(w)\cdot\widehat{f}(w)$ følger av konvolusjonsregelen at

$$h(x) = \mathcal{F}^{-1}\{\widehat{h}(w)\} = \frac{2\pi}{\sqrt{2\pi}}\,(f * f)(x) = \sqrt{2\pi}\int_{-\infty}^{\infty} f(x-p)f(p)\,\mathrm{d}p.$$

Når $p < 0$ er $f(p) = 0$, og når $p > x$, dvs. $x - p < 0$, er $f(x-p) = 0$. Følgelig er $h(x) = \sqrt{2\pi}\int_0^x f(x-p)f(p)\,dp$. Dermed får vi $h(x) = 0$ når $x < 0$ og

$$h(x) = \sqrt{2\pi}\int_0^x \mathrm{e}^{-(x-p)}e^{-p}\,\mathrm{d}p = \sqrt{2\pi}\int_0^x e^{-x}\,\mathrm{d}p = \sqrt{2\pi}\,\mathrm{e}^{-x}p\Big|_{p=0}^{x} = \sqrt{2\pi}\,x\mathrm{e}^{-x} \quad \text{når } x > 0.$$

**2**

$$\text{Invers Fouriertransformert:} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\frac{\cos(\pi w/2)}{1-w^2}e^{iwx}dw = f(x) \text{ for alle } x$$

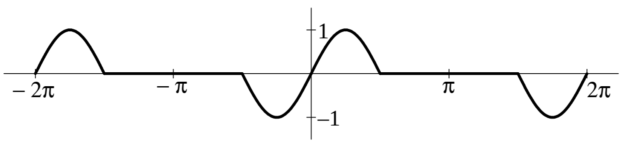
$$x = 0 : \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos(\pi w/2)}{1-w^2}dw = f(0) = 1 \quad \text{dvs.} \quad \int_{-\infty}^{\infty}\frac{\cos(\pi w/2)}{1-w^2}dw = \pi$$

$$x = \pi/2 : \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos(\pi w/2)}{1-w^2}e^{i\pi w/2}dw = f(\pi/2) = 0$$

$$\text{Eulers formel gir} \int_{-\infty}^{\infty}\frac{\cos(\pi w/2)}{1-w^2}\cos\frac{\pi w}{2} + i\int_{-\infty}^{\infty}\frac{\cos(\pi w/2)}{1-w^2}\sin\frac{(\pi w/2)}{2}dw = 0$$

$$\text{og følgelig:} \int_{-\infty}^{\infty}\frac{\cos^2(\pi w/2)}{1-w^2}dw = 0$$

**3** **a)**




$$b_2 = \frac{2}{\pi}\int_0^\pi f(x)\sin 2x\,dx = \frac{2}{\pi}\int_0^{\pi/2}\sin 2x \sin 2x\,dx = \frac{1}{\pi}\int_0^{\pi/2}(1-\cos 4x)\,dx = \frac{1}{2}$$

For $n \neq 2$ er

$$b_n = 0 \quad \text{for } n = 2m \quad \text{og} \quad b_n = -\frac{4}{\pi}\frac{(-1)^m}{(2m-1)(2m+3)} \quad \text{for } n = 2m+1$$

siden $\sin(2m \cdot \pi/2) = 0$ og $\sin[(2m+1)\pi/2] = \sin(m\pi + \pi/2) = \cos m\pi = (-1)^m$. Fourierrekka til $f(x)$ er en sinusrekke, og $f(x)$ er kontinuerlig for alle $x$. Altså har vi

$$\begin{aligned} f(x) &= \frac{1}{2}\sin 2x - \frac{4}{\pi}\sum_{\substack{n=1\\ n\neq 2}}^{\infty} \frac{\sin(n\pi/2)}{(n-2)(n+2)}\sin nx \\ &= \frac{1}{2}\sin 2x - \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m-1)(2m+3)}\sin(2m+1)x \qquad \text{for alle } x. \end{aligned}$$

**b)** For $x = \pi/2$ er $f(x) = 0$ og $\sin(2m+1)x = (-1)^m$. Det gir

$$0 = f\left(\frac{\pi}{2}\right) = \frac{1}{2}\sin\pi - \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{(-1)^m(-1)^m}{(2m-1)(2m+3)} = -\frac{4}{\pi}\sum_{m=0}^{\infty} \frac{1}{(2m-1)(2m+3)}$$

$$\sum_{m=0}^{\infty} \frac{1}{(2m-1)(2m+3)} = \frac{1}{(-1)\cdot 3} + \frac{1}{1\cdot 5} + \frac{1}{3\cdot 7} + \frac{1}{5\cdot 9} + \cdots = 0.$$

For å finne summen av den andre rekka, kan vi bruke Parsevals identitet:

$$\frac{1}{4} + \frac{16}{\pi^2}\sum_{m=0}^{\infty}\left[\frac{(-1)^m}{(2m-1)(2m+3)}\right]^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx = \frac{2}{\pi}\int_0^{\pi/2}\sin^2 2x\,dx = \frac{1}{2}$$

$$\sum_{m=0}^{\infty} \frac{1}{(2m-1)^2(2m+3)^2} = \left(\frac{1}{2} - \frac{1}{4}\right)\Big/\frac{16}{\pi^2} = \frac{\pi^2}{64}$$