



Opgavesettet har 11 punkter, 1ab, 2ab, 3ab, 4ab, 5, 6, 7, som teller likt ved bedømmelsen.

1 a)

$$\frac{1}{(s-1)(s-2)(s-3)} = \frac{1}{2} \cdot \frac{1}{s-1} - \frac{1}{s-2} + \frac{1}{2} \cdot \frac{1}{s-3} \quad (\text{Ved delbrøkkopp spalting.})$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s-2)(s-3)}\right\} = \frac{1}{2}e^t - e^{2t} + \frac{1}{2}e^{3t}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{e^{-as}}{(s-1)(s-2)(s-3)}\right\} = u(t-a)\left[\frac{1}{2}e^{t-a} - e^{2(t-a)} + \frac{1}{2}e^{3(t-a)}\right]$$

b)

$$g(t) = [1 - u(t-2)]4e^{t-2} \cdot e^2 \Rightarrow G(s) = \mathcal{L}(g) = \frac{4}{s-1} - 4e^2 \frac{e^{-2s}}{s-1}$$

(Det kunne vi også ha regnet ut direkte fra definisjonen av Laplacetransformasjonen.)

$$Y(s) = \mathcal{L}(y) \quad \text{gir} \quad \mathcal{L}(y') = sY \quad \text{og} \quad \mathcal{L}(y'') = s^2Y$$

Transformerer ligningen og løser mhp. Y :

$$s^2Y - 5sY + 6Y = \frac{4}{s-1}(1 - e^2 \cdot e^{-2s})$$

$$Y = \frac{4(1 - e^2 e^{-2s})}{(s-1)(s^2 - 5s + 6)} = \frac{4}{(s-1)(s-2)(s-3)} - \frac{4e^2 e^{-2s}}{(s-1)(s-2)(s-3)}$$

Resultatet i **a)** gir

$$y = \mathcal{L}^{-1}(Y) = 2e^t - 4e^{2t} + 2e^{3t} - e^2 u(t-2) [2e^{t-2} - 4e^{2(t-2)} + 2e^{3(t-2)}].$$

2 a)

$$f_1(t) * f_2(t) = \int_0^t \cos(t-\tau) \cdot 2 \cos^2 \tau \, d\tau = \int_0^t \cos(t-\tau)(1 + \cos 2\tau) \, d\tau$$

$$= \int_0^t (\cos(t-\tau) + \cos(t-\tau) \cos 2\tau) \, d\tau$$

$$= \int_0^t (\cos(t-\tau) + \frac{1}{2}[\cos(t+\tau) + \cos(t-3\tau)]) \, d\tau$$

$$= \left[-\sin(t-\tau) + \frac{1}{2} \sin(t+\tau) - \frac{1}{6} \sin(t-3\tau) \right]_0^t = \frac{2}{3}(\sin t + \sin 2t)$$

Eller:

$$\begin{aligned}\mathcal{L}(f_1 * f_2) &= \mathcal{L}(\cos t) \cdot \mathcal{L}(2 \cos^2 t) = \mathcal{L}(\cos t) \cdot \mathcal{L}(1 + \cos 2t) \\ &= \frac{s}{s^2 + 1} \cdot \left(\frac{1}{s} + \frac{s}{s^2 + 4} \right) = \frac{1}{s^2 + 1} + \frac{s^2}{(s^2 + 1)(s^2 + 4)} \\ f_1(t) * f_2(t) &= \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 1} + \frac{1}{3} \left(\frac{4}{s^2 + 4} - \frac{1}{s^2 + 1} \right) \right\} = \frac{2}{3} (\sin t + \sin 2t)\end{aligned}$$

b) Ved å bruke konvolusjonsregelen og tabellen i Rottmann side 168 får vi

$$\mathcal{F}(g_1 * g_2) = \sqrt{2\pi} \frac{1}{\sqrt{2\pi} a} \pi e^{-a|w|} \cdot \frac{1}{\sqrt{2\pi} b} \pi e^{-b|w|} = \frac{1}{\sqrt{2\pi} ab} \pi^2 e^{-(a+b)|w|}.$$

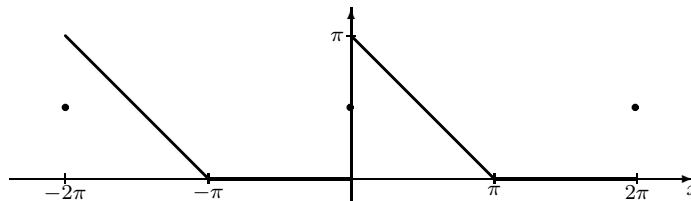
Samme formel i Rottmann gir

$$\begin{aligned}\mathcal{F} \left\{ \frac{1}{(a+b)^2 + x^2} \right\} &= \frac{1}{\sqrt{2\pi} a + b} \pi e^{-(a+b)|w|} = \frac{ab}{\pi(a+b)} \mathcal{F}(g_1 * g_2) \\ \int_{-\infty}^{\infty} \frac{1}{a^2 + p^2} \cdot \frac{1}{b^2 + (x-p)^2} dp &= g_1(x) * g_2(x) = \frac{\pi(a+b)}{ab} \frac{1}{(a+b)^2 + x^2}.\end{aligned}$$

3 a)

$$\begin{aligned}a_0 &= \frac{1}{2\pi} \int_0^\pi (\pi - x) dx = \frac{1}{2\pi} \left[\pi x - \frac{1}{2} x^2 \right]_0^\pi = \frac{1}{4} \pi. \quad (\text{Kan også sees geometrisk.}) \\ \pi a_n &= \int_0^\pi (\pi - x) \cos nx dx = \left[(\pi - x) \frac{\sin nx}{n} \right]_0^\pi + \frac{1}{n} \int_0^\pi \sin nx dx \\ &= -\frac{1}{n^2} \left[\cos nx \right]_0^\pi = \frac{1 - (-1)^n}{n^2}. \quad \text{Altså: } a_{2m} = 0, \quad a_{2m+1} = \frac{2}{\pi(2m+1)^2} \cdot \\ \pi b_n &= \int_0^\pi (\pi - x) \sin nx dx = \left[-(\pi - x) \frac{\cos nx}{n} \right]_0^\pi - \frac{1}{n} \underbrace{\int_0^\pi \cos nx dx}_0 = \frac{\pi}{n}, \quad b_n = \frac{1}{n} \\ f(t) &\sim \frac{\pi}{4} + \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{\cos(2m+1)x}{(2m+1)^2} + \sum_{n=1}^{\infty} \frac{\sin nx}{n}\end{aligned}$$

b)



For $x = \pi/2$ er summen av Fourierrekka lik $f(\pi/2) = \pi/2$. Altså blir

$$\begin{aligned}\frac{\pi}{2} &= \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n} = \frac{\pi}{4} + \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \quad \text{idet } \begin{cases} \sin(2m \cdot \pi/2) = 0 \\ \sin([2m+1] \cdot \pi/2) = (-1)^m \end{cases} \\ \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} &= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.\end{aligned}$$

For $x = 0$ er summen av Fourierrekka lik $\frac{1}{2}[f(0+) + f(0-)] = \pi/2$. Altså blir

$$\frac{\pi}{2} = \frac{\pi}{4} + \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} \quad \text{som gir} \quad \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \frac{\pi}{2} \left(\frac{\pi}{2} - \frac{\pi}{4} \right) = \frac{\pi^2}{8}.$$

4 a) Separasjon av variable:

$$u(x, t) = F(x) \cdot G(t) \text{ innsatt i (1): } F''G = FG'' - 4FG$$

$$\frac{F''}{F} = \frac{G''}{G} - 4 = -\lambda \quad (\text{konstant}) \quad \text{som gir} \quad \begin{cases} F'' + \lambda F = 0 \\ G'' + (\lambda - 4)G = 0 \end{cases}$$

Randbetingelsene $F(0) = 0$ og $F(\pi) = 0$ fører, på vanlig måte, til ikke-trivielle løsninger for $F(x)$ når $\lambda = n^2$, $n = 1, 2, 3, \dots$:

$$F(x) = F_n(x) = \sin nx, \quad n = 1, 2, 3, \dots$$

Setter vi inn $\lambda = n^2$ i ligningen for $G(t)$, får vi $G'' + (n^2 - 4)G = 0$.

$$n = 1: \quad G'' - (\sqrt{3})^2 G = 0$$

$$G_1(t) = A_1 \cosh(\sqrt{3}t) + B_1 \sinh(\sqrt{3}t) \quad (\text{eller } G_1(t) = C_1 e^{\sqrt{3}t} + D_1 e^{-\sqrt{3}t})$$

$$n = 2: \quad G'' = 0, \quad G_2(t) = A_2 + B_2 t$$

$$n = 3, 4, \dots: \quad G'' + (\sqrt{n^2 - 4})^2 G = 0, \quad G_n(t) = A_n \cos(\sqrt{n^2 - 4}t) + B_n \sin(\sqrt{n^2 - 4}t)$$

Produktløsningene blir da:

$$u_1(x, t) = [A_1 \cosh(\sqrt{3}t) + B_1 \sinh(\sqrt{3}t)] \sin x$$

$$u_2(x, t) = [A_2 + B_2 t] \sin 2x$$

$$u_n(x, t) = [A_n \cos(\sqrt{n^2 - 4}t) + B_n \sin(\sqrt{n^2 - 4}t)] \sin nx, \quad n = 3, 4, 5, \dots$$

b) "Generell løsning":

$$u(x, t) = [A_1 \cosh(\sqrt{3}t) + B_1 \sinh(\sqrt{3}t)] \sin x + [A_2 + B_2 t] \sin 2x \\ + \sum_{n=3}^{\infty} [A_n \cos(\sqrt{n^2 - 4}t) + B_n \sin(\sqrt{n^2 - 4}t)] \sin nx$$

Vi bruker initialbetingelsene (2) til å bestemme A_n og B_n for $n = 1, 2, 3, \dots$

$$0 = u(x, 0) = A_1 \sin x + A_2 \sin 2x + \sum_{n=3}^{\infty} A_n \sin nx$$

Dermed er $A_n = 0$ for $n = 1, 2, 3, \dots$, og

$$u(x, t) = B_1 \sinh(\sqrt{3}t) \sin x + B_2 t \sin 2x + \sum_{n=3}^{\infty} B_n \sin(\sqrt{n^2 - 4}t) \sin nx$$

$$u_t(x, t) = \sqrt{3} B_1 \cosh(\sqrt{3}t) \sin x + B_2 \sin 2x + \sum_{n=3}^{\infty} \sqrt{n^2 - 4} B_n \cos(\sqrt{n^2 - 4}t) \sin nx$$

$$\sin x + \sin 2x + \sin 3x = u_t(x, 0) = \sqrt{3} B_1 \sin x + B_2 \sin 2x + \sum_{n=3}^{\infty} \sqrt{n^2 - 4} B_n \sin nx.$$

Dermed er $\sqrt{3}B_1 = 1$, $B_2 = 1$, $\sqrt{3^2 - 4}B_3 = 1$ og $\sqrt{n^2 - 4}B_n = 0$ for $n = 4, 5, 6, \dots$.
 Altså $B_1 = 1/\sqrt{3}$, $B_2 = 1$, $B_3 = 1/\sqrt{5}$ og $B_n = 0$ for $n = 4, 5, 6, \dots$, og løsningen blir

$$u(x, t) = \frac{1}{\sqrt{3}} \sinh(\sqrt{3}t) \sin x + t \sin 2x + \frac{1}{\sqrt{5}} \sin(\sqrt{5}t) \sin 3x.$$

5 Infører nye variabler: $Y_1 = y$, $Y_2 = y'$, $Y_3 = y''$.

$$Y' = \begin{pmatrix} y' \\ y'' \\ y''' \end{pmatrix} = \begin{pmatrix} Y_2 \\ Y_3 \\ 3Y_3 - 6Y_2 + 6Y_1 \end{pmatrix} \quad \text{med starbetingelse } Y(1) = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}$$

Heuns metode med $h = 0.1$:

$$K1 = hF(Y(1)) = \begin{pmatrix} 0.1000 \\ -0.4000 \\ -0.6000 \end{pmatrix}, \quad K2 = hF(Y(1) + K1) = \begin{pmatrix} 0.0600 \\ -0.4600 \\ -0.4800 \end{pmatrix}$$

$$Y(1.1) \approx Y(1) + (K1 + K2)/2 = \begin{pmatrix} 2.0800 \\ 0.5700 \\ -4.5400 \end{pmatrix}$$

Alternativ:

$$K1 = hF(Y(0)) = \begin{pmatrix} -0.1000 \\ 0 \\ 0.5000 \\ 0.2000 \end{pmatrix}, \quad K2 = hF(Y(0) + K1) = \begin{pmatrix} -0.1500 \\ 0.0200 \\ -0.4500 \\ 0.1800 \end{pmatrix}$$

$$Y(0.1) \approx Y(0) + (K1 + K2)/2 = \begin{pmatrix} 0.8750 \\ 0.0100 \\ -1.4750 \\ 0.1900 \end{pmatrix}$$

6 Jacobimatrisen:

$$J(x, y) = \begin{pmatrix} 1 + 13/x & -2y \\ 4x - y - 5 & -x \end{pmatrix}, \quad J(x_0, y_0) = \begin{pmatrix} 3.6 & -10.0 \\ 10.0 & -5.0 \end{pmatrix}$$

$$\begin{pmatrix} 3.6 & -10.0 \\ 10.0 & -5.0 \end{pmatrix} \begin{pmatrix} \Delta x_0 \\ \Delta y_0 \end{pmatrix} = -F(x_0, y_0) = \begin{pmatrix} 0.9227 \\ 1.0000 \end{pmatrix}$$

$$\begin{pmatrix} \Delta x_0 \\ \Delta y_0 \end{pmatrix} = \begin{pmatrix} -0.0657 \\ 0.0686 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} \Delta x_0 \\ \Delta y_0 \end{pmatrix} = \begin{pmatrix} 4.9343 \\ 5.0686 \end{pmatrix}$$

7 En iterasjon med Gauss-Seidel gir:

$$\begin{aligned} x_1^{(1)} &= \frac{1}{6}(11 + 2x_2^{(0)} - x_3^{(0)}) = 2.0833 \\ x_2^{(1)} &= \frac{1}{7}(5 + 2x_1^{(1)} - 2x_3^{(0)}) = 0.8890 \\ x_3^{(1)} &= \frac{1}{-5}(-1 - x_1^{(1)} - 2x_2^{(1)}) = 0.9690 \end{aligned}$$