SIF5013 Matematikk 4N
3. aug 2000

Norges teknisk-naturvitenskapelige universitet
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Løsningsforslag

Oppgavesettet har 11 punkter, 1ab, 2ab, 3ab, 4ab, 5, 6, 7, som teller likt ved bedømmelsen.

$\boxed{1}$ **a)**

$$\frac{1}{(s-1)(s-2)(s-3)} = \frac{1}{2}\cdot\frac{1}{s-1} - \frac{1}{s-2} + \frac{1}{2}\cdot\frac{1}{s-3} \quad \text{(Ved delbrøkoppspalting.)}$$

$$\mathcal{L}^{-1}\Big\{\frac{1}{(s-1)(s-2)(s-3)}\Big\} = \frac{1}{2}e^{t} - e^{2t} + \frac{1}{2}e^{3t}$$

$$f(t) = \mathcal{L}^{-1}\Big\{\frac{e^{-as}}{(s-1)(s-2)(s-3)}\Big\} = u(t-a)\Big[\frac{1}{2}e^{t-a} - e^{2(t-a)} + \frac{1}{2}e^{3(t-a)}\Big]$$

b)

$$g(t) = \big[1-u(t-2)\big]4e^{t-2}\cdot e^{2} \quad\Rightarrow\quad G(s) = \mathcal{L}(g) = \frac{4}{s-1} - 4e^{2}\frac{e^{-2s}}{s-1}$$

(Det kunne vi også ha regnet ut direkte fra definisjonen av Laplacetransformasjonen.)

$$Y(s) = \mathcal{L}(y) \quad \text{gir} \quad \mathcal{L}(y') = sY \text{ og } \mathcal{L}(y'') = s^2Y$$

Transformerer ligningen og løser mhp. Y:

$$s^2Y - 5sY + 6Y = \frac{4}{s-1}\big(1 - e^{2}\cdot e^{-2s}\big)$$

$$Y = \frac{4(1-e^{2}e^{-2s})}{(s-1)(s^2-5s+6)} = \frac{4}{(s-1)(s-2)(s-3)} - \frac{4e^{2}e^{-2s}}{(s-1)(s-2)(s-3)}$$

Resultatet i **a)** gir

$$y = \mathcal{L}^{-1}(Y) = 2e^{t} - 4e^{2t} + 2e^{3t} - e^{2}u(t-2)\big[2e^{t-2} - 4e^{2(t-2)} + 2e^{3(t-2)}\big].$$

$\boxed{2}$ **a)**

$$\begin{aligned}
f_1(t)*f_2(t) &= \int_0^t \cos(t-\tau)\cdot 2\cos^2\tau\,d\tau = \int_0^t \cos(t-\tau)(1+\cos 2\tau)\,d\tau\\
&= \int_0^t \Big(\cos(t-\tau) + \cos(t-\tau)\cos 2\tau\Big)\,d\tau\\
&= \int_0^t \Big(\cos(t-\tau) + \tfrac{1}{2}[\cos(t+\tau) + \cos(t-3\tau)]\Big)\,d\tau\\
&= \Big[-\sin(t-\tau) + \tfrac{1}{2}\sin(t+\tau) - \tfrac{1}{6}\sin(t-3\tau)\Big]_0^t = \tfrac{2}{3}(\sin t + \sin 2t)
\end{aligned}$$

Eller:

$$\begin{aligned}
\mathcal{L}(f_1*f_2) &= \mathcal{L}(\cos t)\cdot\mathcal{L}(2\cos^2 t) = \mathcal{L}(\cos t)\cdot\mathcal{L}(1+\cos 2t)\\
&= \frac{s}{s^2+1}\cdot\Big(\frac{1}{s} + \frac{s}{s^2+4}\Big) = \frac{1}{s^2+1} + \frac{s^2}{(s^2+1)(s^2+4)}\\
f_1(t)*f_2(t) &= \mathcal{L}^{-1}\Big\{\frac{1}{s^2+1} + \frac{1}{3}\Big(\frac{4}{s^2+4} - \frac{1}{s^2+1}\Big)\Big\} = \frac{2}{3}(\sin t + \sin 2t)
\end{aligned}$$

b) Ved å bruke konvolusjonsregelen og tabellen i Rottmann side 168 får vi

$$\mathcal{F}(g_1*g_2) = \sqrt{2\pi}\,\frac{1}{\sqrt{2\pi}}\frac{\pi}{a}e^{-a|w|}\cdot\frac{1}{\sqrt{2\pi}}\frac{\pi}{b}e^{-b|w|} = \frac{1}{\sqrt{2\pi}}\frac{\pi^2}{ab}e^{-(a+b)|w|}.$$

Samme formel i Rottmann gir

$$\mathcal{F}\Big\{\frac{1}{(a+b)^2+x^2}\Big\} = \frac{1}{\sqrt{2\pi}}\frac{\pi}{a+b}e^{-(a+b)|w|} = \frac{ab}{\pi(a+b)}\mathcal{F}(g_1*g_2)$$

$$\int_{-\infty}^{\infty}\frac{1}{a^2+p^2}\cdot\frac{1}{b^2+(x-p)^2}\,dp = g_1(x)*g_2(x) = \frac{\pi(a+b)}{ab}\,\frac{1}{(a+b)^2+x^2}.$$

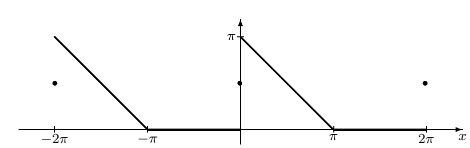
$\boxed{3}$ **a)**

$$a_0 = \frac{1}{2\pi}\int_0^{\pi}(\pi-x)\,dx = \frac{1}{2\pi}\Big[\pi x - \tfrac{1}{2}x^2\Big]_0^{\pi} = \tfrac{1}{4}\pi. \qquad \text{(Kan også sees geometrisk.)}$$

$$\begin{aligned}
\pi a_n &= \int_0^{\pi}(\pi-x)\cos nx\,dx = \Big[(\pi-x)\frac{\sin nx}{n}\Big]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\sin nx\,dx\\
&= -\frac{1}{n^2}\Big[\cos nx\Big]_0^{\pi} == \frac{1-(-1)^n}{n^2}. \quad \text{Altså:} \quad a_{2m} = 0,\ a_{2m+1} = \frac{2}{\pi(2m+1)^2}\,.\\
\pi b_n &= \int_0^{\pi}(\pi-x)\sin nx\,dx = \Big[-(\pi-x)\frac{\cos nx}{n}\Big]_0^{\pi} - \frac{1}{n}\underbrace{\int_0^{\pi}\cos nx\,dx}_{0} = \frac{\pi}{n}\,,\ b_n = \frac{1}{n}
\end{aligned}$$

$$f(t) \sim \frac{\pi}{4} + \frac{2}{\pi}\sum_{m=0}^{\infty}\frac{\cos(2m+1)x}{(2m+1)^2} + \sum_{n=1}^{\infty}\frac{\sin nx}{n}$$

b)



For $x = \pi/2$ er summen av Fourierrekka lik $f(\pi/2) = \pi/2$. Altså blir

$$\frac{\pi}{2} = \frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{\sin(n\pi/2)}{n} = \frac{\pi}{4} + \sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1} \quad \text{idet} \quad \begin{cases}\sin(2m\cdot\pi/2) = 0\\ \sin([2m+1]\cdot\pi/2) = (-1)^m\end{cases}$$

$$\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

For $x = 0$ er summen av Fourierrekka lik $\frac{1}{2}\big[f(0+) + f(0-)\big] = \pi/2$. Altså blir

$$\frac{\pi}{2} = \frac{\pi}{4} + \frac{2}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2} \quad \text{som gir} \quad \sum_{m=0}^{\infty}\frac{1}{(2m+1)^2} = \frac{\pi}{2}\Big(\frac{\pi}{2} - \frac{\pi}{4}\Big) = \frac{\pi^2}{8}.$$

$\boxed{4}$ **a)** Separasjon av variable:

$$u(x,t) = F(x)\cdot G(t) \text{ innsatt i (1)}: \quad F''G = FG'' - 4FG$$

$$\frac{F''}{F} = \frac{G''}{G} - 4 = -\lambda \quad \text{(konstant)} \quad \text{som gir} \quad \begin{cases} F'' + \lambda F = 0 \\ G'' + (\lambda - 4)G = 0 \end{cases}$$

Randbetingelsene $F(0) = 0$ og $F(\pi) = 0$ fører, på vanlig måte, til ikke-trivielle løsninger for $F(x)$ når $\lambda = n^2$, $n = 1,2,3,\dots$:

$$F(x) = F_n(x) = \sin nx, \quad n = 1,2,3,\dots$$

Setter vi inn $\lambda = n^2$ i ligningen for $G(t)$, får vi $G'' + (n^2-4)G = 0$.

$n = 1$: $\quad G'' - (\sqrt{3})^2 G = 0$

$\quad G_1(t) = A_1 \cosh(\sqrt{3}\,t) + B_1 \sinh(\sqrt{3}\,t) \quad$ (eller $G_1(t) = C_1 e^{\sqrt{3}\,t} + D_1 e^{-\sqrt{3}\,t}$)

$n = 2:$ $\quad G'' = 0, \quad G_2(t) = A_2 + B_2 t$

$n = 3,4,\dots:$ $G'' + (\sqrt{n^2-4})^2 G = 0, \quad G_n(t) = A_n \cos(\sqrt{n^2-4}\,t) + B_n \sin(\sqrt{n^2-4}\,t)$

Produktløsningene blir da:

$$u_1(x,t) = \big[A_1\cosh(\sqrt{3}\,t) + B_1\sinh(\sqrt{3}\,t)\big]\sin x$$
$$u_2(x,t) = [A_2 + B_2 t]\sin 2x$$
$$u_n(x,t) = \big[A_n\cos(\sqrt{n^2-4}\,t) + B_n\sin(\sqrt{n^2-4}\,t)\big]\sin nx, \quad n = 3,4,5,\dots$$

b) "Generell løsning":

$$u(x,t) = \big[A_1\cosh(\sqrt{3}\,t) + B_1\sinh(\sqrt{3}\,t)\big]\sin x + [A_2 + B_2 t]\sin 2x + \sum_{n=3}^{\infty}\big[A_n\cos(\sqrt{n^2-4}\,t) + B_n\sin(\sqrt{n^2-4}\,t)\big]\sin nx$$

Vi bruker initialbetingelsene (2) til å bestemme A_n og B_n for $n = 1,2,3\dots$

$$0 = u(x,0) = A_1\sin x + A_2\sin 2x + \sum_{n=3}^{\infty} A_n \sin nx$$

Dermed er $A_n = 0$ for $n = 1,2,3,\dots$, og

$$u(x,t) = B_1\sinh(\sqrt{3}\,t)\sin x + B_2 t\sin 2x + \sum_{n=3}^{\infty} B_n\sin(\sqrt{n^2-4}\,t)\sin nx$$

$$u_t(x,t) = \sqrt{3}\,B_1\cosh(\sqrt{3}\,t)\sin x + B_2\sin 2x + \sum_{n=3}^{\infty}\sqrt{n^2-4}\,B_n\cos(\sqrt{n^2-4}\,t)\sin nx$$

$$\sin x + \sin 2x + \sin 3x = u_t(x,0) = \sqrt{3}\,B_1\sin x + B_2\sin 2x + \sum_{n=3}^{\infty}\sqrt{n^2-4}\,B_n\sin nx.$$

Dermed er $\sqrt{3}\,B_1 = 1$, $B_2 = 1$, $\sqrt{3^2-4}\,B_3 = 1$ og $\sqrt{n^2-4}\,B_n = 0$ for $n = 4,5,6,\dots$. Altså $B_1 = 1/\sqrt{3}$, $B_2 = 1$, $B_3 = 1/\sqrt{5}$ og $B_n = 0$ for $n = 4,5,6,\dots$, og løsningen blir

$$u(x,t) = \frac{1}{\sqrt{3}}\sinh(\sqrt{3}\,t)\sin x + t\sin 2x + \frac{1}{\sqrt{5}}\sin(\sqrt{5}\,t)\sin 3x.$$

$\boxed{5}$ Infører nye variabler: $Y_1 = y$, $Y_2 = y'$, $Y_3 = y''$.

$$Y' = \begin{pmatrix} y' \\ y'' \\ y''' \end{pmatrix} = \begin{pmatrix} Y_2 \\ Y_3 \\ 3Y_3 - 6Y_2 + 6Y_1 \end{pmatrix} \qquad \text{med starbetingelse } Y(1) = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}$$

Heuns metode med $h = 0.1$:

$$K1 = hF(Y(1)) = \begin{pmatrix} 0.1000 \\ -0.4000 \\ -0.6000 \end{pmatrix}, \qquad K2 = hF(Y(1) + K1) = \begin{pmatrix} 0.0600 \\ -0.4600 \\ -0.4800 \end{pmatrix}$$

$$Y(1.1) \approx Y(1) + (K1 + K2)/2 = \begin{pmatrix} 2.0800 \\ 0.5700 \\ -4.5400 \end{pmatrix}$$

Alternativ:

$$K1 = hF(Y(0)) = \begin{pmatrix} -0.1000 \\ 0 \\ 0.5000 \\ 0.2000 \end{pmatrix}, \qquad K2 = hF(Y(0) + K1) = \begin{pmatrix} -0.1500 \\ 0.0200 \\ -0.4500 \\ 0.1800 \end{pmatrix}$$

$$Y(0.1) \approx Y(0) + (K1 + K2)/2 = \begin{pmatrix} 0.8750 \\ 0.0100 \\ -1.4750 \\ 0.1900 \end{pmatrix}$$

$\boxed{6}$ Jacobimatrisen:

$$J(x,y) = \begin{pmatrix} 1 + 13/x & -2y \\ 4x - y - 5 & -x \end{pmatrix}, \qquad J(x_0,y_0) = \begin{pmatrix} 3.6 & -10.0 \\ 10.0 & -5.0 \end{pmatrix}$$

$$\begin{pmatrix} 3.6 & -10.0 \\ 10.0 & -5.0 \end{pmatrix}\begin{pmatrix} \Delta x_0 \\ \Delta y_0 \end{pmatrix} = -F(x_0,y_0) = \begin{pmatrix} 0.9227 \\ 1.0000 \end{pmatrix}$$

$$\begin{pmatrix} \Delta x_0 \\ \Delta y_0 \end{pmatrix} = \begin{pmatrix} -0.0657 \\ 0.0686 \end{pmatrix}, \qquad \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} \Delta x_0 \\ \Delta y_0 \end{pmatrix} = \begin{pmatrix} 4.9343 \\ 5.0686 \end{pmatrix}$$

$\boxed{7}$ En iterasjon med Gauss-Seidel gir:

$$\begin{aligned} x_1^{(1)} &= \tfrac{1}{6}(11 + 2x_2^{(0)} - x_3^{(0)}) &&= 2.0833 \\ x_2^{(1)} &= \tfrac{1}{7}(5 + 2x_1^{(1)} - 2x_3^{(0)}) &&= 0.8890 \\ x_3^{(1)} &= \tfrac{1}{-5}(-1 - x_1^{(1)} - 2x_2^{(1)}) &&= 0.9690 \end{aligned}$$