



Oppgavesettet har 12 punkter, 1abc, 2abc, 3ab, 4, 5, 6, 7, som teller likt ved bedømmelsen.

1 a) $g(t) = (t - \pi)u(t - \pi) = f_1(t - \pi)u(t - \pi), \quad f_1(t) = t$

$$G(s) = \mathcal{L}(g) = F_1(s)e^{-\pi s} = \frac{1}{s^2}e^{-\pi s} \quad \text{ved skiftteorem 2, eller, alternativt,}$$

$$G(s) = \int_0^\infty g(t)e^{-st} dt = \int_\pi^\infty (t - \pi)e^{-st} dt = \left[-\frac{t - \pi}{s}e^{-st} - \frac{1}{s^2}e^{-st} \right]_\pi^\infty = \frac{1}{s^2}e^{-\pi s}$$

$$H(s) = \frac{s + 1}{(s + 1)^2 + 1} = F_2(s + 1), \quad F_2(s) = \frac{s}{s^2 + 1}$$

$$h(t) = \mathcal{L}^{-1}(H) = e^{-t}f_2(t) = e^{-t} \cos t \quad \text{ved skiftteorem 1}$$

b) $Y(s) = \mathcal{L}\{y(t)\}, \quad y(t) + t * y(t) = g(t) \Rightarrow Y + \frac{1}{s^2} \cdot Y = \frac{1}{s^2}e^{-\pi s}$

$$Y(s) = \frac{s^2}{s^2 + 1} \cdot \frac{1}{s^2}e^{-\pi s} = \frac{1}{s^2 + 1}e^{-\pi s}$$

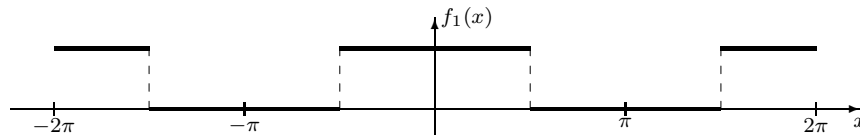
$$y(t) = \mathcal{L}^{-1}(Y) = \sin(t - \pi)u(t - \pi) = -u(t - \pi) \sin t$$

c) $Y(s) = \mathcal{L}\{y(t)\}, \quad s^2Y - s - (-1) + 2(sY - 1) + 2Y = e^{-\pi s}$

$$Y(s) = \frac{s + 1}{s^2 + 2s + 2} + \frac{1}{s^2 + 2s + 2}e^{-\pi s} = H(s) + \frac{1}{(s + 1)^2 + 1}e^{-\pi s}$$

$$y(t) = \mathcal{L}^{-1}(Y) = h(t) + [e^{-(t-\pi)} \sin(t - \pi)]u(t - \pi) = e^{-t} \cos t - e^{\pi-t}u(t - \pi) \sin t$$

2 a)



$$a_0 = \frac{1}{\pi} \int_0^\pi f(x) dx = \frac{1}{\pi} \int_0^{\pi/2} dx = \frac{1}{2}$$

$$a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx dx = \frac{2}{\pi} \int_0^{\pi/2} \cos nx dx = \frac{2 \sin nx}{\pi n\pi} \Big|_0^{\pi/2} = \frac{2 \sin(n\pi/2)}{n\pi}$$

$$a_n = 0 \text{ for } n \text{ partall, } a_n = \frac{2}{n\pi} \text{ for } n = 1, 5, \dots, a_n = \frac{-2}{n\pi} \text{ for } n = 3, 7, \dots$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \left(\cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - + \dots \right) = \frac{1}{2} + \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m \cos (2m+1)x}{2m+1}$$

b)

$u(x, t) = F(x) \cdot G(t)$ innsettes i (1):

$$F''G = FG + FG', \quad \frac{F''}{F} = 1 + \frac{G'}{G} = k \quad (\text{konstant})$$

$$(I) \quad F'' - kF = 0, \quad F'(0) = 0, \quad F'(\pi) = 0, \quad (II) \quad G' - (k-1)G = 0$$

$$(I) \quad k > 0, \quad k = \mu^2: \quad F(x) = Ae^{\mu x} + Be^{-\mu x}, \quad F'(x) = \mu Ae^{\mu x} - \mu Be^{-\mu x}$$

$$F'(0) = F'(\pi) = 0 \Rightarrow A = B = 0, \quad F(x) = 0$$

$$k = 0: \quad F(x) = A + Bx, \quad F'(x) = B$$

$$F'(0) = F'(\pi) = 0 \Rightarrow B = 0, \quad F(x) = 1, \quad (A = 1)$$

$$k < 0, \quad k = -p^2: \quad F(x) = A \cos px + B \sin px, \quad F'(x) = -pA \sin px + pB \cos px$$

$$F'(0) = 0 \Rightarrow B = 0, \quad F'(\pi) = 0, \quad A \neq 0 \Rightarrow p = n, \quad F(x) = \cos nx, \quad (A = 1)$$

$$F(x) = \cos nx, \quad n = 0, 1, 2, 3, \dots$$

$$(II) \quad k = -n^2, \quad n = 0, 1, 2, \dots: \quad G' + (n^2 + 1)G = 0, \quad G(t) = Ce^{-(n^2+1)t}$$

Løsningene av (1) på formen $u(x, t) = F(x)G(t)$ som tilfredsstill (2):

$$u_n(x, t) = C_n e^{-(n^2+1)t} \cos nx, \quad C_n \text{ vilkårlig konstant, } n = 0, 1, 2, \dots$$

c) Superposisjonsprinsippet:

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) = \sum_{n=0}^{\infty} C_n e^{-(n^2+1)t} \cos nx \quad \text{oppfyller (1) og (2)}$$

$$f(x) = u(x, 0) = \sum_{n=0}^{\infty} C_n \cos nx \Rightarrow C_n = a_n \stackrel{\text{a)}}{=} \begin{cases} 1/2 & \text{for } n = 0 \\ 2/(n\pi) & \text{for } n = 1, 5, \dots \\ -2/(n\pi) & \text{for } n = 3, 7, \dots \end{cases}$$

$$u(x, t) = \frac{1}{2}e^{-t} + \frac{2}{\pi} \left(e^{-2t} \cos x - \frac{1}{3}e^{-(3^2+1)t} \cos 3x + \frac{1}{5}e^{-(5^2+1)t} \cos 5x - + \dots \right) \\ = \frac{1}{2}e^{-t} + \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} e^{-[(2m+1)^2+1]t} \cos (2m+1)x, \quad 0 \leq x \leq \pi, \quad t > 0$$

Omformer: $2 \cos x \cdot \cos 2x = \cos (x - 2x) + \cos (x + 2x) = \cos x + \cos 3x$

$$\cos x + \cos 3x = u(x, 0) = \sum_{n=0}^{\infty} C_n \cos nx \Rightarrow C_1 = 1, \quad C_3 = 1, \quad C_n = 0 \text{ ellers}$$

$$u(x, t) = e^{-(1^2+1)t} \cos x + e^{-(3^2+1)t} \cos 3x = e^{-2t} \cos x + e^{-10t} \cos 3x$$

3 a) $x = 0 :$

$$\frac{2 \sinh \pi}{\pi} \left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} \right] = f(0) = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} = \frac{\pi}{2 \sinh \pi} - \frac{1}{2}$$

 $x = \pi :$

$$\frac{2 \sinh \pi}{\pi} \left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} (-1)^n \right] = \frac{f(\pi+0) + f(\pi-0)}{2} = \frac{e^\pi + e^{-\pi}}{2} = \cosh \pi$$

Følgelig: $\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{1+n^2} = \frac{\pi}{2 \sinh \pi} \cdot \cosh \pi$ dvs. $\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{\pi \cosh \pi}{2 \sinh \pi} - \frac{1}{2}$

b)

Invers Fouriertransformert: $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{\cos(\pi w/2)}{1-w^2} e^{iwx} dw = f(x)$ for alle x

$$x = 0 : \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos(\pi w/2)}{1-w^2} dw = f(0) = 1 \quad \text{dvs.} \quad \int_{-\infty}^{\infty} \frac{\cos(\pi w/2)}{1-w^2} dw = \pi$$

$$x = \pi/2 : \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos(\pi w/2)}{1-w^2} e^{i\pi w/2} dw = f(\pi/2) = 0$$

Eulers formel gir $\int_{-\infty}^{\infty} \frac{\cos(\pi w/2)}{1-w^2} \cos \frac{\pi w}{2} dw + i \underbrace{\int_{-\infty}^{\infty} \frac{\cos(\pi w/2)}{1-w^2} \sin \frac{\pi w}{2} dw}_0 = 0$

og følgelig: $\int_{-\infty}^{\infty} \frac{\cos^2(\pi w/2)}{1-w^2} dw = 0$

4 Gitt er tre punkter, altså er polynomet av grad to,

$$p_2(x) = \sum_{j=0}^2 f_j L_j(x) = L_1(x)$$

$$L_1(x) = \frac{(x-0)(x-\pi)}{(\pi/2-0)(\pi/2-\pi)} = -4 \frac{x(x-\pi)}{\pi^2} .$$

Punktene er jevnt fordelt, altså er

$$|\sin(x) - p_2(x)| \leq \frac{1}{4(2+1)} \left(\frac{\pi}{2}\right)^3 = 0.3230 .$$

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$$T_{\pi/8} = \frac{\pi}{16} (\tan(0) + 2 \tan(\pi/8) + \tan(\pi/4)) = \frac{\pi}{16} (2\sqrt{2} - 1) = 0.3590$$

$$\left| T_h - \int_0^{\pi/4} \tan(x) dx \right| \leq \frac{M_2}{12} \frac{(b-a)^3}{n^2} = \frac{M_2}{12} (b-a) h^2 , \quad h = \frac{b-a}{n} .$$

Vi har

$$f(x) = \tan(x) , \quad f'(x) = \frac{1}{\cos^2(x)} , \quad f''(x) = 2 \frac{\sin(x)}{\cos^3(x)} .$$

Telleren i $f''(x)$ vokser og nevneren avtar i $[0, \pi/4]$. Dermed er $f''(x)$ voksende, og

$$|f''(x)| \leq M_2 = 2 \frac{\sin(\pi/4)}{\cos^3(\pi/4)} = 4 .$$

Betingelsen

$$\left| T_h - \int_0^{\pi/4} \tan(x) dx \right| \leq \frac{M_2 \pi}{12} h^2 \leq 10^{-4}$$

oppfylles altså med

$$h \leq \sqrt{\frac{12}{\pi}} 10^{-2} = 1.9544 \cdot 10^{-2} < 0.02 .$$

6 Ved innføring av nye variable

$$Y_1 = x_1 , \quad Y_2 = x_2 , \quad Y_3 = x'_1 , \quad Y_4 = x'_2$$

finner vi at

$$Y' = \begin{pmatrix} x'_1 \\ x'_2 \\ -\frac{k_1}{m_1} x_1 + \frac{k_2}{m_1} (x_2 - x_1) - \frac{d}{m_1} x'_1 \\ -\frac{k_2}{m_2} (x_2 - x_1) - \frac{k_3}{m_2} x_2 - \frac{d}{m_2} x'_2 \end{pmatrix} ,$$

altså

$$Y' = \begin{pmatrix} Y_3 \\ Y_4 \\ -\frac{k_1}{m_1} Y_1 + \frac{k_2}{m_1} (Y_2 - Y_1) - \frac{d}{m_1} Y_3 \\ -\frac{k_2}{m_2} (Y_2 - Y_1) - \frac{k_3}{m_2} Y_2 - \frac{d}{m_2} Y_4 \end{pmatrix} =: F(Y)$$

Startbetingelsene blir da

$$Y(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \\ x'_1(0) \\ x'_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} .$$

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$$K_1 = hF(Y(0)) = 0.1 \begin{pmatrix} 2 - 2 \cdot 2 \\ 2 \cdot 2 - 2 \end{pmatrix} = \begin{pmatrix} -0.2 \\ 0.2 \end{pmatrix} ,$$

$$K_2 = hF(Y(0) + K_1) = 0.1F \begin{pmatrix} 1.8 \\ 2.2 \end{pmatrix} = 0.1 \begin{pmatrix} 1.8 - 1.8 \cdot 2.2 \\ 1.8 \cdot 2.2 - 2.2 \end{pmatrix} = \begin{pmatrix} -0.216 \\ 0.176 \end{pmatrix} ,$$

$$Y(0.1) \approx Y(0) + \frac{1}{2}(K_1 + K_2) = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -0.416 \\ 0.376 \end{pmatrix} = \begin{pmatrix} 1.792 \\ 2.188 \end{pmatrix} .$$