



Oppgavesettet har 10 punkter, 1ab, 2ab, 3ab, 4, 5abc, som teller likt ved bedømmelsen.

1 a) Alternativ 1:

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{b^2}{s-b}\right) &= b^2 e^{bt} \Rightarrow \mathcal{L}^{-1}\left(\frac{b^2}{s(s-b)}\right) = \int_0^t b^2 e^{b\tau} d\tau = b(e^{bt} - 1) \Rightarrow \\ \mathcal{L}^{-1}\left(\frac{b^2}{s^2(s-b)}\right) &= \int_0^t b(e^{b\tau} - 1) d\tau = [e^{b\tau} - b\tau]_0^t = e^{bt} - bt - 1\end{aligned}$$

Dermed er

$$\mathcal{L}^{-1}\{F(s)\} = 2 \sinh bt + e^{bt} - bt - 1 = 2e^{bt} - e^{-bt} - bt - 1$$

Alternativ 2: delbrøkkopp spalting.

$$F(s) = \frac{A}{s-b} + \frac{B}{s+b} + \frac{C_1}{s} + \frac{C_2}{s^2}, \quad A = 2, B = -1, C_1 = -1, C_2 = -b$$

som gir samme resultat som ovenfor. Ved skiftteorem 2 får vi

$$\mathcal{L}^{-1}\{H(s)\} = f(t-a)u(t-a) = [2e^{b(t-a)} - e^{-b(t-a)} - b(t-a) - 1]u(t-a).$$

b) Vi Laplacetransformerer først $g(t) = tu(t-1)$ ved skiftteorem 2 og $\mathcal{L}(t^n) = n!/s^{n+1}$.

$$g(t) = [(t-1) + 1]u(t-1) \Rightarrow \mathcal{L}(g) = \left(\frac{1}{s^2} + \frac{1}{s}\right)e^{-s} = \frac{s+1}{s^2}e^{-s}$$

(Det kunne vi også ha regnet ut direkte fra definisjonen av Laplacetransformasjonen.)

Laplacetransformerer differensialligningen og løser mhp. $Y = \mathcal{L}(y)$:

$$\begin{aligned}s^2 Y - sy(0) - y'(0) - Y &= 2e^{-s} + \frac{s+1}{s^2}e^{-s} \quad (y(0) = 1, y'(0) = 1) \\ Y &= \frac{s+1}{s^2-1} + \left(\frac{2}{s^2-1} + \frac{s+1}{s^2(s^2-1)}\right)e^{-s} = \frac{1}{s-1} + \left(\frac{2}{s^2-1} + \frac{1}{s^2(s-1)}\right)e^{-s}\end{aligned}$$

Resultatene i a) (med $a = b = 1$) gir

$$y = e^t + [2e^{t-1} - e^{-(t-1)} - t]u(t-1) = \begin{cases} e^t & \text{for } t < 1 \\ e^t + 2e^{t-1} - e^{-(t-1)} - t & \text{for } t > 1. \end{cases}$$

Vi ser at $y'(t) = e^t$ for $t < 1$, dermed har $y(t)$ venstrederivert e i $t = 1$. For $t > 1$ får vi $y'(t) = e^t + 2e^{t-1} + e^{-(t-1)} - 1$. Da har $y(t)$ høyrederivert $e + 2$ for $t = 1$, og $y(t)$ er følgelig *ikke* deriverbar for $t = 1$.

2 a) Vi setter $u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} B_n e^{-n^2 c^2 t} \sin nx$ og bestemmer B_n -ene.

$$\begin{aligned} f(x) \stackrel{\text{(iii)}}{=} u(x, 0) &= \sum_{n=1}^{\infty} B_n \sin nx \quad (\text{for } 0 \leq x \leq \pi) \Rightarrow B_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \\ B_n &= \frac{2}{\pi} \left(\int_0^{\pi/2} x \sin nx \, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin nx \, dx \right) \\ &= \frac{2}{\pi} \left(\left[-x \frac{\cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^{\pi/2} + \left[-(\pi - x) \frac{\cos nx}{n} - \frac{\sin nx}{n^2} \right]_{\pi/2}^{\pi} \right) \\ &= \frac{2}{\pi} \left(\left[-\frac{\pi}{2} \frac{\cos(n\pi/2)}{n} + \frac{\sin(n\pi/2)}{n^2} \right] + \left[\frac{\pi}{2} \frac{\cos(n\pi/2)}{n} + \frac{\sin(n\pi/2)}{n^2} \right] \right) \\ &= \frac{4 \sin(n\pi/2)}{\pi n^2} \end{aligned}$$

Dermed får vi

$$\begin{aligned} B_n &= 0 \quad \text{når } n = 2m \quad \text{og} \quad B_n = \frac{4}{\pi} \frac{(-1)^m}{(2m+1)^2} \quad \text{når } n = 2m+1 \\ u(x, t) &= \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} e^{-(2m+1)^2 c^2 t} \sin(2m+1)x \quad \text{oppfyller (i), (ii) og (iii)} \end{aligned}$$

b) Vi setter inn $u(x, t) = F(x)G(t)$ i (i) og bruker randbetingelsen (iv).

$$\begin{aligned} FG' &= c^2 F''G \Rightarrow \frac{F''}{F} = \frac{G'}{c^2 G} = k \quad (\text{konstant}) \\ \text{(I)} \quad F'' - kF &= 0, \quad F(0) = 0, \quad F'(\pi) = 0, \quad \text{(II)} \quad G' - kc^2 G = 0 \end{aligned}$$

Bestemmer først $F(x)$ og deretter $G(t)$.

$$\begin{aligned} \text{(I)} \quad F'' - kF &= 0, \quad F(0) = 0, \quad F'(\pi) = 0 \\ k > 0, \quad k = \mu^2: \quad F(x) &= Ae^{\mu x} + Be^{-\mu x}, \quad F'(x) = \mu Ae^{\mu x} - \mu Be^{-\mu x} \\ F(0) = F'(\pi) = 0 &\Rightarrow A = B = 0, \quad F(x) = 0 \\ k = 0: \quad F(x) &= A + Bx, \quad F'(x) = B \\ F(0) = 0 \Rightarrow A = 0, \quad F'(\pi) = 0 &\Rightarrow B = 0, \quad F(x) = 0 \\ k < 0, \quad k = -p^2: \quad F(x) &= A \cos px + B \sin px, \quad F'(x) = -pA \sin px + pB \cos px \\ F(0) = 0 \Rightarrow A = 0, \quad F'(\pi) = 0, \quad B \neq 0 &\Rightarrow \cos p\pi = 0 \\ p = (2m+1)/2, \quad F(x) &= \sin[(2m+1)x/2] \quad (B = 1) \\ \text{(II)} \quad G' - kc^2 G &= 0 \\ k = -\left(\frac{2m+1}{2}\right)^2 &\Rightarrow G' + \frac{c^2(2m+1)^2}{4} G = 0 \Rightarrow G(t) = Ce^{-c^2(2m+1)^2 t/4} \end{aligned}$$

Løsningene av (i) på formen $u(x, t) = F(x)G(t)$ som tilfredsstill (iv):

$$u(x, t) = Ce^{-c^2(2m+1)^2 t/4} \sin \frac{(2m+1)x}{2}, \quad C \text{ vilkårlig konstant, } m = 0, 1, 2, \dots$$

3 a) Fouriertransformerer $u_{xx} = tu_t$.

(Bruker betingelsene $\lim_{|x| \rightarrow \infty} u(x, t) = \lim_{|x| \rightarrow \infty} u_x(x, t) = 0$ når u_{xx} transformeres.)

$$-w^2 \hat{u} = t \frac{\partial \hat{u}}{\partial t} \quad \text{dvs.} \quad \frac{1}{\hat{u}} \frac{\partial \hat{u}}{\partial t} = -\frac{w^2}{t}, \quad \text{separabel differensialligning}$$

$$\ln |\hat{u}| = -w^2 \ln t + C_1(w) \Rightarrow \hat{u}(w, t) = C(w) e^{-w^2 \ln t} \quad (C(w) = \pm e^{C_1(w)})$$

$$\left. \begin{aligned} \hat{u}(w, 1) &= C(w) e^{w^2 \ln 1} = C(w) \\ \hat{u}(w, 1) &= \mathcal{F}\{u(x, 1)\} = \hat{f}(w) \end{aligned} \right\} \Rightarrow C(w) = \hat{f}(w), \quad \hat{u}(w, t) = \hat{f}(w) e^{-w^2 \ln t}$$

Løsningen kan også skrives $\hat{u}(w, t) = \hat{f}(w) t^{-w^2}$.

b) Fra tabell i Rottmann:

$$\mathcal{F}\left(e^{-ax^2}\right) = \frac{1}{\sqrt{2\pi}} \cdot \sqrt{\frac{\pi}{a}} e^{-w^2/4a} = \frac{1}{\sqrt{2a}} e^{-w^2/4a}, \quad \text{setter } a = \frac{1}{4 \ln A}$$

$$\hat{g}_A(w) = \mathcal{F}\left(e^{-x^2/4 \ln A}\right) = \frac{1}{\sqrt{1/(2 \ln A)}} e^{-w^2 \ln A} = \sqrt{2 \ln A} e^{-w^2 \ln A}$$

Vi bruker uttrykket for $\hat{g}_A(w)$ med $A = t$ til å skrive $e^{-w^2 \ln t}$ som en Fouriertransformert, og finner deretter $u(x, t)$ ved å bruke konvolusjonsteoremet.

$$e^{-w^2 \ln t} = \mathcal{F}\left(\frac{1}{\sqrt{2 \ln t}} e^{-x^2/4 \ln t}\right) = \frac{1}{\sqrt{2 \ln t}} \mathcal{F}\{h(x, t)\}, \quad h(x, t) = e^{-x^2/4 \ln t}$$

$$\hat{u}(w, t) = \hat{f}(w) e^{-w^2 \ln t} = \frac{1}{\sqrt{2 \ln t}} \frac{1}{\sqrt{2\pi}} \left(\sqrt{2\pi} \mathcal{F}(f) \mathcal{F}(h)\right)$$

$$u(x, t) = \frac{1}{2\sqrt{\pi \ln t}} f(x) * h(x, t) = \frac{1}{2\sqrt{\pi \ln t}} \int_{-\infty}^{\infty} f(x-p) h(p, t) dp$$

Vi kan altså skrive $u(x, t)$ på den oppgitte formen og $h(x, t) = e^{-x^2/4 \ln t}$.

4 Fra den komplekse Fourierrekka kan vi finne den vanlige Fourierrekka.

$$\begin{aligned} f(x) &\sim \frac{\sinh \pi}{\pi} \left[\sum_{n=-1}^{-\infty} (-1)^n \frac{1+in}{1+n^2} e^{inx} + 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1+in}{1+n^2} e^{inx} \right] \\ &= \frac{\sinh \pi}{\pi} \left[\sum_{n=1}^{\infty} (-1)^n \frac{1-in}{1+(-n)^2} e^{-inx} + 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1+in}{1+n^2} e^{inx} \right] \\ &= \frac{\sinh \pi}{\pi} \left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{(1-in)e^{-inx} + (1+in)e^{inx}}{1+n^2} \right] \\ &= \frac{2 \sinh \pi}{\pi} \left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} (\cos nx - n \sin nx) \right] \end{aligned}$$

Setter inn $x = 0$:

$$1 = \frac{2 \sinh \pi}{\pi} \left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} (1-0) \right] \quad \text{som gir} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} = \frac{\pi}{2 \sinh \pi} - \frac{1}{2}$$

Setter inn $x = \pi$ og bruker at $\lim_{x \rightarrow \pi^-} f(x) = e^\pi$, $\lim_{x \rightarrow \pi^+} f(x) = e^{-\pi}$:

$$\frac{e^\pi + e^{-\pi}}{2} = \frac{2 \sinh \pi}{\pi} \left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} \cdot (-1)^n \right] \quad \text{som gir} \quad \sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{\pi \cosh \pi}{2 \sinh \pi} - \frac{1}{2}$$

5 a) Ligningene for de 3 indre nodene er

$$\begin{aligned} u_1^{n+1} - u_1^n &= \frac{k}{2h^2} (u_2^n - 2u_1^n + u_2^{n+1} - 2n_1^{n+1}) \\ u_2^{n+1} - u_2^n &= \frac{k}{2h^2} (u_3^n - 2u_2^n + u_1^n + u_3^{n+1} - 2n_2^{n+1} + u_1^{n+1}) \\ u_3^{n+1} - u_3^n &= \frac{k}{2h^2} (-2u_3^n + u_2^n - 2n_3^{n+1} + u_2^{n+1}) \end{aligned}$$

La $n = 0$, $\frac{k}{2h^2} = \frac{1}{2}$, $u_1^0 = u(\frac{1}{4}, 0) = \frac{3}{2}$, $u_2^0 = u(\frac{1}{2}, 0) = 2$, $u_3^0 = u(\frac{3}{4}, 0) = \frac{3}{2}$. Dette gir

$$\begin{aligned} 2u_1^1 - \frac{1}{2}u_2^1 &= 1 & 4u_1^1 - u_2^1 &= 2 \\ -\frac{1}{2}u_1^1 + 2u_2^1 - \frac{1}{2}u_3^1 &= \frac{3}{2} & \text{eller} \quad -u_1^1 + 4u_2^1 - u_3^1 &= 3 \\ -\frac{1}{2}u_2^1 + 2u_3^1 &= 1 & -u_2^1 + 4u_3^1 &= 2 \end{aligned}$$

b) Gauss-Seidel-metoden blir

$$\begin{aligned} x_{n+1} &= \frac{1}{4}y_n + \frac{1}{2} \\ y_{n+1} &= \frac{1}{4}x_{n+1} + \frac{1}{4}z_n + \frac{3}{4} \\ z_{n+1} &= \frac{1}{4}y_{n+1} + \frac{1}{2} \end{aligned}$$

La $x_0 = \frac{3}{4}$, $y_0 = 1$, $z_0 = \frac{3}{4}$. Dette gir

$$x_1 = \frac{3}{4}, \quad y_1 = \frac{9}{8}, \quad z_1 = \frac{25}{32}.$$

c) Lagrange-interpolasjonspolynomet er

$$\begin{aligned} p(x) &= 0 \cdot \frac{(x-1)(x-2)(x-3)(x-4)}{(-1)(-2)(-3)(-4)} + 11 \cdot \frac{x(x-2)(x-3)(x-4)}{1 \cdot (-1)(-2)(-3)} \\ &+ 16 \cdot \frac{x(x-1)(x-3)(x-4)}{2 \cdot 1 \cdot (-1)(-2)} + 11 \cdot \frac{x(x-1)(x-2)(x-4)}{3 \cdot 2 \cdot 1 \cdot (-1)} \\ &+ 0 \cdot \frac{x(x-1)(x-2)(x-3)}{4 \cdot 3 \cdot 2 \cdot 1}. \end{aligned}$$

og dette gir

$$p(x) = \frac{x(x-4)}{3}(x^2 - 4x - 8) = \frac{1}{3}(x^4 - 8x^3 + 8x^2 + 32x).$$