SIF5016 Matematikk 4N
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Norges teknisk-naturvitenskapelige universitet
Institutt for matematiske fag

Løsningsforslag

Oppgavesettet har 10 punkter, 1ab, 2ab, 3ab, 4, 5abc, som teller likt ved bedømmelsen.

1 **a)** Alternativ 1:

$$\mathcal{L}^{-1}\left(\frac{b^2}{s-b}\right) = b^2 e^{bt} \;\Rightarrow\; \mathcal{L}^{-1}\left(\frac{b^2}{s(s-b)}\right) = \int_0^t b^2 e^{b\tau}\,d\tau = b(e^{bt}-1) \;\Rightarrow$$

$$\mathcal{L}^{-1}\left(\frac{b^2}{s^2(s-b)}\right) = \int_0^t b(e^{b\tau}-1)\,d\tau = \Big[e^{b\tau} - b\tau\Big]_0^t = e^{bt} - bt - 1$$

Dermed er

$$\mathcal{L}^{-1}\{F(s)\} = 2\sinh bt + e^{bt} - bt - 1 = 2e^{bt} - e^{-bt} - bt - 1$$

Alternativ 2: delbrøkoppspalting.

$$F(s) = \frac{A}{s-b} + \frac{B}{s+b} + \frac{C_1}{s} + \frac{C_2}{s^2}, \qquad A = 2,\ B = -1,\ C_1 = -1,\ C_2 = -b$$

som gir samme resultat som ovenfor. Ved skiftteorem 2 får vi

$$\mathcal{L}^{-1}\{H(s)\} = f(t-a)u(t-a) = \Big[2e^{b(t-a)} - e^{-b(t-a)} - b(t-a) - 1\Big]\,u(t-a).$$

b) Vi Laplacetransformerer først $g(t) = tu(t-1)$ ved skiftteorem 2 og $\mathcal{L}(t^n) = n!/s^{n+1}$.

$$g(t) = [(t-1)+1]\,u(t-1) \;\Rightarrow\; \mathcal{L}(g) = \left(\frac{1}{s^2} + \frac{1}{s}\right)e^{-s} = \frac{s+1}{s^2}e^{-s}$$

(Det kunne vi også ha regnet ut direkte fra definisjonen av Laplacetransformasjonen.)
Laplaceransformerer differensialligningen og løser mhp. $Y = \mathcal{L}(y)$:

$$s^2Y - sy(0) - y'(0) - Y = 2e^{-s} + \frac{s+1}{s^2}e^{-s} \qquad (y(0) = 1,\ y'(0) = 1)$$

$$Y = \frac{s+1}{s^2-1} + \left(\frac{2}{s^2-1} + \frac{s+1}{s^2(s^2-1)}\right)e^{-s} = \frac{1}{s-1} + \left(\frac{2}{s^2-1} + \frac{1}{s^2(s-1)}\right)e^{-s}$$

Resultatene i **a)** (med $a = b = 1$) gir

$$y = e^t + \Big[2e^{t-1} - e^{-(t-1)} - t\Big]\,u(t-1) = \begin{cases} e^t & \text{for } t < 1 \\ e^t + 2e^{t-1} - e^{-(t-1)} - t & \text{for } t > 1. \end{cases}$$

Vi ser at $y'(t) = e^t$ for $t < 1$, dermed har $y(t)$ venstrederivert e i $t = 1$. For $t > 1$ får vi $y'(t) = e^t + 2e^{t-1} + e^{-(t-1)} - 1$. Da har $y(t)$ høyrederivert $e + 2$ for $t = 1$, og $y(t)$ er følgelig *ikke* deriverbar for $t = 1$.

2 **a)** Vi setter $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2c^2t}\sin nx$ og bestemmer B_n-ene.

$$f(x) \overset{\text{(iii)}}{=} u(x,0) = \sum_{n=1}^{\infty} B_n \sin nx \ (\text{for } 0 \le x \le \pi) \;\Rightarrow\; B_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$\begin{aligned}
B_n &= \frac{2}{\pi}\left(\int_0^{\pi/2} x\sin nx\,dx + \int_{\pi/2}^{\pi}(\pi - x)\sin nx\,dx\right) \\
&= \frac{2}{\pi}\left(\left[-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi/2} + \left[-(\pi - x)\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right]_{\pi/2}^{\pi}\right) \\
&= \frac{2}{\pi}\left(\left[-\frac{\pi}{2}\frac{\cos(n\pi/2)}{n} + \frac{\sin(n\pi/2)}{n^2}\right] + \left[\frac{\pi}{2}\frac{\cos(n\pi/2)}{n} + \frac{\sin(n\pi/2)}{n^2}\right]\right) \\
&= \frac{4}{\pi}\frac{\sin(n\pi/2)}{n^2}
\end{aligned}$$

Dermed får vi

$$B_n = 0 \quad \text{når } n = 2m \quad \text{og} \quad B_n = \frac{4}{\pi}\frac{(-1)^m}{(2m+1)^2} \quad \text{når } n = 2m+1$$

$$u(x,t) = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^2}e^{-(2m+1)^2c^2t}\sin(2m+1)x \qquad \text{oppfyller (i), (ii) og (iii)}$$

b) Vi setter inn $u(x,t) = F(x)G(t)$ i (i) og bruker randbetingelsen (iv).

$$FG' = c^2F''G \;\Rightarrow\; \frac{F''}{F} = \frac{G'}{c^2G} = k \quad (\text{konstant})$$

$$\text{(I)}\quad F'' - kF = 0,\quad F(0) = 0,\ F'(\pi) = 0, \qquad \text{(II)}\quad G' - kc^2G = 0$$

Bestemmer først $F(x)$ og deretter $G(t)$.

$$\begin{aligned}
&\text{(I)}\quad F'' - kF = 0,\ F(0) = 0,\ F'(\pi) = 0 \\
&k > 0,\ k = \mu^2: \quad F(x) = Ae^{\mu x} + Be^{-\mu x},\ F'(x) = \mu Ae^{\mu x} - \mu Be^{-\mu x} \\
&\qquad F(0) = F'(\pi) = 0 \Rightarrow A = B = 0,\ F(x) = 0 \\
&k = 0: \quad F(x) = A + Bx,\ F'(x) = B \\
&\qquad F(0) = 0 \Rightarrow A = 0, \quad F'(\pi) = 0 \Rightarrow B = 0,\ F(x) = 0 \\
&k < 0,\ k = -p^2: \quad F(x) = A\cos px + B\sin px,\ F'(x) = -pA\sin px + pB\cos px \\
&\qquad F(0) = 0 \Rightarrow A = 0, \quad F'(\pi) = 0,\ B \ne 0 \Rightarrow \cos p\pi = 0 \\
&\qquad p = (2m+1)/2, \quad F(x) = \sin[(2m+1)x/2] \quad (B = 1) \\
&\text{(II)}\quad G' - kc^2G = 0 \\
&k = -\left(\frac{2m+1}{2}\right)^2 \;\Rightarrow\; G' + \frac{c^2(2m+1)^2}{4}G = 0 \;\Rightarrow\; G(t) = Ce^{-c^2(2m+1)^2t/4}
\end{aligned}$$

Løsningene av (i) på formen $u(x,t) = F(x)G(t)$ som tilfredsstiller (iv):

$$u(x,t) = Ce^{-c^2(2m+1)^2t/4}\sin\frac{(2m+1)x}{2}, \quad C \text{ vilkårlig konstant},\ m = 0,\ 1,\ 2,\ \ldots$$

3 **a)** Fouriertransformerer $u_{xx} = tu_t$.
(Bruker betingelsene $\lim_{|x|\to\infty} u(x,t) = \lim_{|x|\to\infty} u_x(x,t) = 0$ når u_{xx} transformeres.)

$$-w^2\widehat{u} = t\frac{\partial\widehat{u}}{\partial t} \quad \text{dvs.} \quad \frac{1}{\widehat{u}}\frac{\partial\widehat{u}}{\partial t} = -\frac{w^2}{t}, \quad \text{separabel differensialligning}$$

$$\ln|\widehat{u}| = -w^2\ln t + C_1(w) \Rightarrow \widehat{u}(w,t) = C(w)e^{-w^2\ln t} \quad \left(C(w) = \pm e^{C_1(w)}\right)$$

$$\left.\begin{array}{l}\widehat{u}(w,1) = C(w)e^{w^2\ln 1} = C(w)\\ \widehat{u}(w,1) = \mathcal{F}\{u(x,1)\} = \widehat{f}(w)\end{array}\right\} \Rightarrow C(w) = \widehat{f}(w), \quad \widehat{u}(w,t) = \widehat{f}(w)e^{-w^2\ln t}$$

Løsningen kan også skrives $\widehat{u}(w,t) = \widehat{f}(w)\,t^{-w^2}$.

b) Fra tabell i Rottmann:

$$\mathcal{F}\left(e^{-ax^2}\right) = \frac{1}{\sqrt{2\pi}}\cdot\sqrt{\frac{\pi}{a}}\,e^{-w^2/4a} = \frac{1}{\sqrt{2a}}\,e^{-w^2/4a}, \qquad \text{setter } a = \frac{1}{4\ln A}$$

$$\widehat{g}_A(w) = \mathcal{F}\left(e^{-x^2/4\ln A}\right) = \frac{1}{\sqrt{1/(2\ln A)}}\,e^{-w^2\ln A} = \sqrt{2\ln A}\,e^{-w^2\ln A}$$

Vi bruker uttrykket for $\widehat{g}_A(w)$ med $A = t$ til å skrive $e^{-w^2\ln t}$ som en Fouriertransformert, og finner deretter $u(x,t)$ ved å bruke konvolusjonsteoremet.

$$e^{-w^2\ln t} = \mathcal{F}\left(\frac{1}{\sqrt{2\ln t}}e^{-x^2/4\ln t}\right) = \frac{1}{\sqrt{2\ln t}}\,\mathcal{F}\left\{h(x,t)\right\}, \quad h(x,t) = e^{-x^2/4\ln t}$$

$$\widehat{u}(w,t) = \widehat{f}(w)e^{-w^2\ln t} = \frac{1}{\sqrt{2\ln t}}\,\frac{1}{\sqrt{2\pi}}\left(\sqrt{2\pi}\,\mathcal{F}(f)\mathcal{F}(h)\right)$$

$$u(x,t) = \frac{1}{2\sqrt{\pi\ln t}}\,f(x)*h(x,t) = \frac{1}{2\sqrt{\pi\ln t}}\int_{-\infty}^{\infty} f(x-p)h(p,t)\,dp$$

Vi kan altså skrive $u(x,t)$ på den oppgitte formen og $h(x,t) = e^{-x^2/4\ln t}$.

4 Fra den komplekse Fourierrekka kan vi finne den vanlige Fourierrekka.

$$\begin{aligned} f(x) &\sim \frac{\sinh\pi}{\pi}\left[\sum_{n=-1}^{-\infty}(-1)^n\frac{1+in}{1+n^2}e^{inx} + 1 + \sum_{n=1}^{\infty}(-1)^n\frac{1+in}{1+n^2}e^{inx}\right]\\ &= \frac{\sinh\pi}{\pi}\left[\sum_{n=1}^{\infty}(-1)^n\frac{1-in}{1+(-n)^2}e^{-inx} + 1 + \sum_{n=1}^{\infty}(-1)^n\frac{1+in}{1+n^2}e^{inx}\right]\\ &= \frac{\sinh\pi}{\pi}\left[1 + \sum_{n=1}^{\infty}(-1)^n\frac{(1-in)e^{-inx} + (1+in)e^{inx}}{1+n^2}\right]\\ &= \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\left(\cos nx - n\sin nx\right)\right] \end{aligned}$$

Setter inn $x = 0$:

$$1 = \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(1-0)\right] \quad \text{som gir} \quad \sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2} = \frac{\pi}{2\sinh\pi} - \frac{1}{2}$$

Setter inn $x = \pi$ og bruker at $\lim_{x\to\pi-} f(x) = e^{\pi}$, $\lim_{x\to\pi+} f(x) = e^{-\pi}$:

$$\frac{e^{\pi} + e^{-\pi}}{2} = \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\cdot(-1)^n\right] \quad \text{som gir} \quad \sum_{n=1}^{\infty}\frac{1}{1+n^2} = \frac{\pi\cosh\pi}{2\sinh\pi} - \frac{1}{2}$$

5 **a)** Ligningene for de 3 indre nodene er

$$\begin{aligned} u_1^{n+1} - u_1^n &= \frac{k}{2h^2}\left(u_2^n - 2u_1^n + u_2^{n+1} - 2n_1^{n+1}\right)\\ u_2^{n+1} - u_2^n &= \frac{k}{2h^2}\left(u_3^n - 2u_2^n + u_1^n + u_3^{n+1} - 2n_2^{n+1} + u_1^{n+1}\right)\\ u_3^{n+1} - u_3^n &= \frac{k}{2h^2}\left(-2u_3^n + u_2^n - 2n_3^{n+1} + u_2^{n+1}\right) \end{aligned}$$

La $n = 0$, $\frac{k}{2h^2} = \frac{1}{2}$, $u_1^0 = u(\frac{1}{4},0) = \frac{3}{2}$, $u_2^0 = u(\frac{1}{2},0) = 2$, $u_3^0 = u(\frac{3}{4},0) = \frac{3}{2}$. Dette gir

$$\begin{aligned} 2u_1^1 - \frac{1}{2}u_2^1 &= 1\\ -\frac{1}{2}u_1^1 + 2u_2^1 - \frac{1}{2}u_3^1 &= \frac{3}{2}\\ -\frac{1}{2}u_2^1 + 2u_3^1 &= 1 \end{aligned} \qquad \text{eller} \qquad \begin{aligned} 4u_1^1 - u_2^1 &= 2\\ -u_1^1 + 4u_2^1 - u_3^1 &= 3\\ -u_2^1 + 4u_3^1 &= 2 \end{aligned}$$

b) Gauss-Seidel-metoden blir

$$\begin{aligned} x_{n+1} &= \frac{1}{4}y_n + \frac{1}{2}\\ y_{n+1} &= \frac{1}{4}x_{n+1} + \frac{1}{4}z_n + \frac{3}{4}\\ z_{n+1} &= \frac{1}{4}y_{n+1} + \frac{1}{2} \end{aligned}$$

La $x_0 = \frac{3}{4}$, $y_0 = 1$, $z_0 = \frac{3}{4}$. Dette gir

$$x_1 = \frac{3}{4}, \quad y_1 = \frac{9}{8}, \quad z_1 = \frac{25}{32}.$$

c) Lagrange-interpolasjonspolynomet er

$$\begin{aligned} p(x) \;=\; & 0\cdot\frac{(x-1)(x-2)(x-3)(x-4)}{(-1)(-2)(-3)(-4)} + 11\cdot\frac{x(x-2)(x-3)(x-4)}{1\cdot(-1)(-2)(-3)}\\ & +16\cdot\frac{x(x-1)(x-3)(x-4)}{2\cdot 1\cdot(-1)(-2)} + 11\cdot\frac{x(x-1)(x-2)(x-4)}{3\cdot 2\cdot 1\cdot(-1)}\\ & +0\cdot\frac{x(x-1)(x-2)(x-3)}{4\cdot 3\cdot 2\cdot 1\cdot} \end{aligned}$$

og dette gir

$$p(x) = \frac{x(x-4)}{3}(x^2 - 4x - 8) = \frac{1}{3}(x^4 - 8x^3 + 8x^2 + 32x).$$