Løsningsforslag TMA4125
mandag 6. juni 2005

(I)

a. Find the inverse Laplace transform

$$F(s) = \frac{s+4}{(s+2)^2}$$

$$F(s) = \frac{s+2}{(s+2)^2} + \frac{2}{(s+2)^2} = \frac{1}{(s+2)} + \frac{2}{(s+2)^2} \Rightarrow$$

$$(\mathcal{L}^{-1}F)(t) = e^{-2t}(1+2t)$$

b. Solve the initial value problem

$$y''(t) + 4y'(t) + 4y(t) = 0, \quad t \geq 0, \quad y(0) = 1, \quad y'(0) = 0.$$

$$Y(s) := (\mathcal{L}y)(s) \Rightarrow (\mathcal{L}y')(s) = sY(s) - 1$$

$$(\mathcal{L}y'')(s) = s^2 Y(s) - s$$

The Laplace transform of the whole eq-n:

$$s^2 Y(s) - s + 4sY(s) - 4 + 4Y(s) = 0$$

$$\Rightarrow Y(s)(s^2+4s+4) = s+4 \Rightarrow Y(s) = \frac{s+4}{(s+2)^2}$$

$$\Rightarrow y(t) = e^{-2t}(1+2t)$$

c. Solve the integral equation:

$$y(t) + \int_0^t e^{-2(t-\tau)} y(\tau)\, d\tau = e^{-2t}, \quad t > 0$$

Denote $f(t) = e^{-2t}$. The eq-n takes the form

$$y + f * y = f \Rightarrow$$

$$Y(s)(1 + F(s)) = F(s); \text{ where } Y = \mathcal{L}y$$

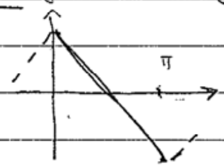
$$F(s) = (\mathcal{L}f)(s) = \frac{1}{s+2}$$

$$Y(s)\left(1 + \frac{1}{s+2}\right) = \frac{1}{s+2} \Rightarrow Y(s) = \frac{1}{s+3}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left(\frac{1}{s+3}\right) = e^{-3t}$$

(II)

a. Find the Fourier series for 2π-periodic function f such that $f(x) = \frac{\pi}{2} - x$, $0 < x < \pi$



$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos kx,$$

since f is even only $\cos kx$ participate in expansion

$a_0 = 0$

$k \geq 1$:

$$a_k = \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2} - x\right)\cos kx\,dx =$$

$$\int_0^{\pi}\cos kx\,dx - \frac{2}{\pi}\int_0^{\pi} x\cos kx\,dx$$

$\int_0^{\pi}\cos kx\,dx = 0$ for all $k \geq 1$.

$$-\frac{2}{\pi}\int_0^{\pi} x\cos kx\,dx = -\frac{2}{\pi k}\int_0^{\pi} x(\sin kx)'\,dx =$$

$$= \underbrace{-\frac{2}{\pi k}\, x\sin kx\Big|_{x=0}^{\pi}}_{=0} + \frac{2}{\pi k}\int_0^{\pi}\sin kx\,dx =$$

$$= -\frac{2}{\pi k^2}\cos kx\Big|_0^{\pi} = -\frac{2}{\pi k^2}\left((-1)^k - 1\right) = \begin{cases} 0 & k\text{-even} \\ \frac{4}{\pi k^2}, & k\text{-odd} \end{cases}$$

Finally
$$f(x) = \frac{4}{\pi}\sum_{l=0}^{\infty}\frac{1}{(2l+1)^2}\cos(2l+1)x$$

b. Find all solutions of the form $u(x,t) = X(x)T(t)$ for the problem:

(i) $\dfrac{\partial^2 u}{\partial x^2} - 2u - \dfrac{\partial u}{\partial t} = 0$, $0 < x < \pi$, $t > 0$

(ii) $u_x(0,t) = 0$, $u_x(\pi,t) = 0$, $t > 0$

$$\left.\begin{array}{l} u(x,t) = X(x)T(t) \\ (i) \end{array}\right\} \Rightarrow$$

$X''T - 2XT - X\dot{T} = 0 \Rightarrow$ (separation of variables)

$$\frac{X''(x)}{X(x)} = \frac{\dot{T}(t)}{T(t)} + 2 = k$$

$$\Rightarrow \left.\begin{array}{l} X''(x) - kX(x) = 0 \\ \dot{T}(t) - (k-2)T(t) = 0 \end{array}\right\} \text{Two ordinary diff. eq-s.}$$

The problem for $X(x)$

$$\begin{cases} X''(x) - kX(x) = 0, & 0 < x < \pi \\ X'(0) = 0, \quad X'(\pi) = 0 \end{cases}$$

The standard analysis shows:

$k_n = -n^2$, $n = 0, 1, \ldots$; $X_n(x) = A_n \cos nx$, $n = 0, 1, \ldots$

The corresponding eq-n for $T(t)$ takes now the form

$$\dot{T}_n(t) + (n^2+2)\,T_n(t) = 0, \quad n = 0, 1, \ldots$$

$$\Rightarrow T_n(t) = B_n e^{-(n^2+2)t}, \quad n = 0, 1, \ldots$$

The solutions of the form

$$u(x,t) = X(x)\,T(t)$$

are

$$u_n(x,t) = a_n\, e^{-(n^2+2)t} \cos nx, \quad n = 0, 1, \ldots$$

c. Find solution to the problem (i), (ii) part b with (additional) initial condition: $u(x,0) = \frac{\pi}{2} - x, \quad 0 < x < \pi.$

$$u(x,t) = \frac{4}{\pi} \sum_{l=0}^{\infty} \frac{1}{(2l+1)^2}\, e^{-[(2l+1)^2+2]t} \cos(2l+1)x$$

III Find the complex Fourier transform of the function $f(x) = e^{-|x|}$ and then find the value of the integral

$$\int_{-\infty}^{\infty} \frac{\cos\omega}{1+\omega^2}\, d\omega$$

Fourier transform:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx =$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-x(1+i\omega)}\, dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-x(-1+i\omega)}\, dx$$

$$\frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-x(1+i\omega)}\, dx = \frac{1}{\sqrt{2\pi}} \frac{-1}{(1+i\omega)} e^{-x(1+i\omega)} \Big|_0^{\infty} =$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{1+i\omega}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-x(-1+i\omega)}\, dx = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{x(-1+i\omega)}\, dx =$$

$$= \frac{-1}{\sqrt{2\pi}} \frac{1}{-1+i\omega}$$

Finally

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1+i\omega} + \frac{1}{1-i\omega} \right) =$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{1+\omega^2}$$

Inverse Fourier transform:

$$e^{-|x|} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega x}\, d\omega = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega x}}{1+\omega^2}\, d\omega$$

$x = 1$:

$$\frac{\pi}{e} = \int_{-\infty}^{\infty} \frac{e^{i\omega}}{1+\omega^2}\, d\omega = \int_{-\infty}^{\infty} \frac{\cos\omega}{1+\omega^2}\, d\omega + i \underbrace{\int_{-\infty}^{\infty} \frac{\sin\omega}{1+\omega^2}\, d\omega}_{=\,0}$$

0 since $\sin\omega$ is an odd f-n

Finally:

$$\int_{-\infty}^{\infty} \frac{\cos\omega}{1+\omega^2}\, d\omega = \frac{\pi}{e}$$

IV

a. Find the polynomial of the smallest possible degree, solving the interpolation problem:

x_k	-2	-1	0	1	2
$p(x_k)$	6	0	0	0	15

Since p vanishes at $0, \pm 1$ it has the form

$$p(x) = x(x^2-1)\,q(x), \quad q\text{- polynomial of degree } 1.$$

$$p(2) = 15 \Rightarrow 6q(2) = 15 \Rightarrow q(2) = 5/2$$

$$p(-2) = 6 \Rightarrow -6q(-2) = 6 \Rightarrow q(-2) = -1$$

Interpolation problem for $q(x)$:

x:	-2	2
q:	-1	$15/6$

$$\Rightarrow q(x) = \frac{5}{2} + \frac{7}{8}(x-2) = \frac{7}{8}x + \frac{3}{4}$$

$$p(x) = (x^3 - x)\left(\frac{7}{8}x + \frac{3}{4}\right) = \boxed{\frac{7}{8}x^4 + \frac{3}{4}x^3 - \frac{7}{8}x^2 - \frac{3}{4}x}$$

Comment: You could use direct Lagrange ~~or~~ or Newton interpolation. It just would take more time, but so far you did it correct you get full credit.

b. Using the Simpson method with step 1 evaluate $\int_{-2}^{2} p(x)\,dx$

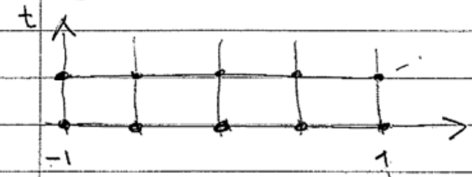
$$I \sim \frac{1}{3}\left[p(-2) + 4p(-1) + 2p(0) + 4p(1) + p(2)\right] =$$

$$= 7$$

V

$$\left.\begin{array}{l} u_t = u_{xx}, \quad -1 \le x \le 1, \quad t > 0 \\ u(-1,t) = 0, \quad u(1,t) = 0, \quad t > 0 \\ u(x,0) = 1 - x^2, \quad -1 \le x \le 1 \end{array}\right\}$$

Using Crank-Nikolson method with the steps $k = 0.5$, $h = 0.5$ write down the system of linear equations for $u_{11} \sim u(-0.5, 0.5)$, $u_{21} \sim u(0, 0.5)$, $u_{31} \sim u(0.5, 0.5)$.



All variables

$u_{00} \sim u(-1,0)$, $u_{1,0} \sim u(-0.5,0)$, $u_{2,0} \sim u(0,0)$

$u_{3,0} \sim u(0.5,0)$, $u_{4,0} \sim u(1,0)$

$u_{0,1} \sim u(-1,0.5)$, $u_{1,1} \sim u(-0.5,0.5)$, $u_{2,1} \sim u(0,0.5)$

$u_{3,1} \sim u(0.5,0.5)$, $u_{4,1} \sim u(1,0.5)$

Boundary and initial conditions $\Rightarrow$

$$\begin{cases} u_{00}=0, \; u_{1,0}=0.75, \; u_{2,0}=1, \; u_{3,0}=0.75, \; u_{4,0}=0 \\ u_{0,1}=0, \; u_{4,1}=0 \end{cases}$$

Auxilliary parameter: $r = \frac{k}{h^2} = 2$

Crank-Nikolson formula for $r=2$

$$(*) \quad 6u_{i,j+1} - 2(u_{i+1,j+1} + u_{i-1,j+1}) = -2u_{ij} + 2(u_{i+1,j} + u_{i-1,j})$$

In our case $j=0$.

$\underline{\underline{i=1}}$: $6u_{1,1} - 2(u_{21} + u_{0,1}) = -2u_{10} + 2(u_{20} + u_{00})$

$\Rightarrow 6u_{11} - 2u_{21} = -1.5 + 4 = 2.5$

$\underline{\underline{i=2}}$: $6u_{2,1} - 2(u_{3,1} + u_{1,1}) = -2u_{20} + 2(u_{3,0} + u_{1,0})$

$\Rightarrow 6u_{21} - 2(u_{3,1} + u_{11}) = -2 + 3.5 = 1.5$

$\underline{\underline{i=3}}$: $6u_{31} - 2(u_{21} + u_{41}) = -2u_{30} + 2(u_{20} + u_{4,0})$

$\Rightarrow 6u_{31} - 2u_{21} = -1.5 + 4 = 2.5$

Write all this together:

$$\begin{cases} 6u_{11} - 2u_{21} = 2.5 \\ -2u_{11} + 6u_{21} - 2u_{31} = 1.5 \\ -2u_{21} + 6u_{31} = 2.5 \end{cases}$$

Comment: By my mistake, when copying the Crank-Nikolson formula to the formula page, I copied the formula which corresponds to $r=1$. Those who applied this formula correctly will receive the full credit.