(I)

a. Find the inverse Laplace transform

$$F(s) = \frac{s+4}{(s+2)^2}$$

$$F(s) = \frac{s+2}{(s+2)^2} + \frac{2}{(s+2)^2} = \frac{1}{(s+2)} + \frac{2}{(s+2)^2} \Rightarrow$$

$$(\mathcal{L}^{-1}F)(t) = e^{-2t}(1+2t)$$

b. Solve the initial value problem
$$y''(t) + 4y'(t) + 4y(t) = 0, \quad t > 0, \quad y(0) = 1, \quad y'(0) = 0.$$

$$Y(s) := (\mathcal{L}y)(s) \Rightarrow (\mathcal{L}y')(s) = sY(s) - 1$$
$$(\mathcal{L}y'')(s) = s^2 Y(s) - s$$

The Laplace transform of the whole eqn:

$$s^2 Y(s) - s + 4sY(s) - 4 + 4Y(s) = 0$$

$$\Rightarrow Y(s)(s^2 + 4s + 4) = s+4 \Rightarrow Y(s) = \frac{s+4}{(s+2)^2}$$

$$\Rightarrow y(t) = e^{-2t}(1+2t)$$

c. Solve the integral equation:

$$y(t) + \int_0^t e^{-2(t-\tau)} y(\tau)\, d\tau = e^{-2t}, \quad t > 0$$

Denote $f(t) = e^{-2t}$. The eq-n takes the form

$$y + f * y = f \quad \Rightarrow$$

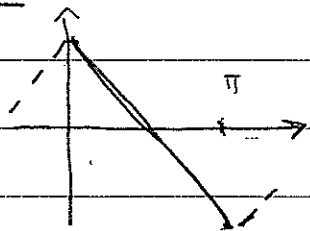$$Y(s)(1 + F(s)) = F(s); \text{ where } Y = \mathcal{L}y$$

$$F(s) = (\mathcal{L}f)(s) = \frac{1}{s+2}$$

$$Y(s)\left(1 + \frac{1}{s+2}\right) = \frac{1}{s+2} \Rightarrow Y(s) = \frac{1}{s+3}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left(\frac{1}{s+3}\right) = e^{-3t}$$

(II)

a. Find the Fourier series for 2π-periodic function f such that $f(x) = \frac{\pi}{2} - x$, $0 < x < \pi$

$$f(x) \sim a_0 + \sum_{k=1}^{\infty} a_k \cos kx,$$

since f is even only $\cos kx$ participate in expansion

$$a_0 = 0$$

$k \geqslant 1:$

$$a_k = \frac{2}{\pi} \int_0^\pi \left(\frac{\pi}{2} - x\right) \cos kx \, dx =$$

$$\int_0^\pi \cos kx \, dx - \frac{2}{\pi} \int_0^\pi x \cos kx \, dx$$

$$\int_0^\pi \cos kx \, dx = 0 \quad \text{for all} \quad k \geqslant 1.$$

$$-\frac{2}{\pi} \int_0^\pi x \cos kx \, dx = -\frac{2}{\pi k} \int_0^\pi x (\sin kx)' \, dx =$$

$$= -\frac{2}{\pi k} x \sin kx \Big|_{x=0}^{\pi} + \frac{2}{\pi k} \int_0^\pi \sin kx \, dx =$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{=0}$$

$$= -\frac{2}{\pi k^2} \cos kx \Big|_0^\pi = -\frac{2}{\pi k^2} \left((-1)^k - 1\right) = \begin{cases} 0 & k\text{-even} \\ \dfrac{4}{\pi k^2}, & k\text{-odd} \end{cases}$$

Finally

$$\boxed{f(x) = \frac{4}{\pi} \sum_{l=0}^{\infty} \frac{1}{(2l+1)^2} \cos(2l+1) x}$$

b. Find all solutions of the form
$u(x,t) = X(x) T(t)$ for the problem:

(i) $\dfrac{\partial^2 u}{\partial x^2} - 2u - \dfrac{\partial u}{\partial t} = 0$, $\quad 0 \leqslant x < \pi$, $\quad t > 0$

(ii) $u_x(0, t) = 0$, $\quad u_x(\pi, t) = 0$, $\quad t > 0$

$$u(x,t) = X(x)\,T(t) \Big\} \Rightarrow$$

(i)

$$X''T - 2XT - X\dot{T} = 0 \quad \Rightarrow \quad \left(\begin{array}{c}\text{separation of}\\\text{variables}\end{array}\right)$$

$$\frac{X''(x)}{X(x)} = \frac{\dot{T}(t)}{T(t)} + 2 = k$$

$$\Rightarrow X''(x) - k\,X(x) = 0 \quad \Big\} \quad \begin{array}{l}\text{Two ordinary}\\ \text{diff. eq-s.}\end{array}$$

$$\dot{T}(t) - (k-2)\,T(t) = 0 \quad \Big\}$$

The problem for $X(x)$

$$\begin{cases} X''(x) - k\,X(x) = 0, & 0 < x < \pi \\ X'(0) = 0, & X'(\pi) = 0 \end{cases}$$

The standard analys shows:

$$k_n = -n^2, \quad n = 0, 1, \dots \; ; \quad X_n(x) = A_n \cos n x,$$
$$n = 0, 1, \dots$$

The corresponding eq-n for $T(t)$ takes now the form

$$\dot{T}_n(t) + (n^2+2)\,T_n(t) = 0, \quad n = 0, 1, \dots$$

$$\Rightarrow T_n(t) = B_n\, e^{-(n^2+2)t}, \quad n = 0, 1, \dots$$

The solutions of the form
$$u(x,t) = X(x) T(t)$$

are

$$u_n(x,t) = a_n e^{-(n^2+2)t} \cos nx, \quad n = 0, 1, \ldots$$

c. Find solution to the problem (i), (ii) part b with (additional) initial condition: $u(x, 0) = \frac{\pi}{2} - x$, $0 < x < \pi$.

$$u(x,t) = \frac{4}{\pi} \sum_{l=0}^{\infty} \frac{1}{(2l+1)^2} e^{-\left[(2l+1)^2+2\right]t} \cos(2l+1)x$$

(III) Find the complex Fourier transform of the function $f(x) = e^{-|x|}$ and then find the value of the integral

$$\int_{-\infty}^{\infty} \frac{\cos \omega}{1+\omega^2} d\omega$$

Fourier transform:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-x(1+i\omega)} dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-x(-1+i\omega)} dx$$

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-x(1+i\omega)} \, dx = \frac{1}{\sqrt{2\pi}} \frac{-1}{(1+i\omega)} e^{-x(1+i\omega)} \Big|_0^\infty = -(-$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{1+i\omega}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{-x(-1+i\omega)} \, dx = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{x(-1+i\omega)} \, dx =$$

$$= \frac{-1}{\sqrt{2\pi}} \frac{1}{-1+i\omega}$$

Finally

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1+i\omega} + \frac{1}{1-i\omega} \right) =$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{1+\omega^2}$$

Inverse Fourier transform:

$$e^{-|x|} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \hat{f}(\omega) e^{i\omega x} \, d\omega = \frac{1}{\pi} \int_{-\infty}^\infty \frac{e^{i\omega x}}{1+\omega^2} \, d\omega$$

$$x = 1 \ :$$

$$\frac{\pi}{e} = \int_{-\infty}^\infty \frac{e^{i\omega}}{1+\omega^2} \, d\omega = \int_{-\infty}^\infty \frac{\cos\omega}{1+\omega^2} \, d\omega + i \int_{-\infty}^\infty \frac{\sin\omega}{1+\omega^2} \, d\omega$$

$$\| $$

$$0 \text{ since } \sin\omega$$
$$\text{is an odd } f\text{—}$$

Finally:

$$\int_{-\infty}^\infty \frac{\cos\omega}{1+\omega^2} \, d\omega = \frac{\pi}{e}$$

IV

a. Find the polynomial of the smallest possible degree, solving the interpolation problem:

x_k	-2	-1	0	1	2
$p(x_k)$	6	0	0	0	15

Since p vanishes at $0, \pm 1$ it has the form

$$p(x) = x(x^2-1)\, q(x), \quad q\text{-polynomial of degree 1.}$$

$p(2) = 15 \Rightarrow 6q(2) = 15 \Rightarrow q(2) = 5/2$

$p(-2) = 6 \Rightarrow -6q(-2) = 6 \Rightarrow q(-2) = -1$

Interpolation problem for $q(x)$:

$x:$	-2	2
$q:$	-1	15/6

$$\Rightarrow q(x) = \frac{5}{2} + \frac{7}{8}(x-2) = \frac{7}{8}x + \frac{3}{4}$$

$$p(x) = (x^3 - x)\left(\frac{7}{8}x + \frac{3}{4}\right) = \boxed{\frac{7}{8}x^4 + \frac{3}{4}x^3 - \frac{7}{8}x^2 - \frac{3}{4}x}$$

Comment: You could use direct Lagrange or Newton interpolation. It just would take more time, but so far you did it correct you get full credit.

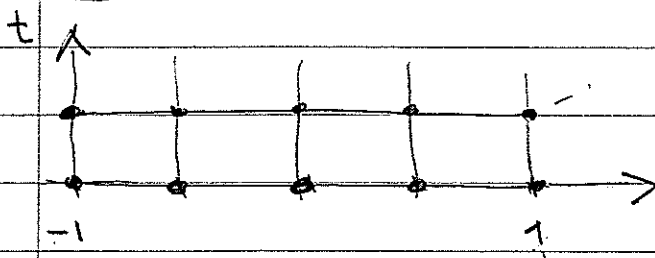6. Using the Simpson method with step 1 evaluate

$$\int_{-2}^{2} p(x)\, dx$$

$$I \sim \frac{1}{3}\left[p(-2) + 4p(-1) + 2p(0) + 4p(1) + p(2) \right] =$$

$$= 7$$

$$\boxed{7} \qquad u_t = u_{xx}, \quad -1 \leq x \leq 1, \quad t > 0$$

$$u(-1, t) = 0, \quad u(1, t) = 0, \quad t > 0$$

$$u(x, 0) = 1 - x^2, \quad -1 \leq x \leq 1$$

using Crank-Nikolson method with the steps $k = 0.5$, $h = 0.5$ write down the system of linear equations for

$u_{11} \sim u(-0.5, 0.5)$, $u_{21} \sim u(0, 0.5)$, $u_{31} \sim u(0.5, 0.5)$

all variables

$u_{00} \sim u(-1, 0)$, $u_{1,0} \sim u(-0.5, 0)$, $u_{2,0} \sim u(0, 0)$

$u_{3,0} \sim u(0.5, 0)$, $u_{4,0} \sim u(1, 0)$

$u_{0,1} \sim u(-1, 0.5)$, $u_{1,1} \sim u(-0.5, 0.5)$, $u_{2,1} \sim u(0, 0.5)$

$u_{3,1} \sim u(0.5, 0.5)$, $u_{4,1} \sim u(1, 0.5)$

Boundary and initial conditions $\Rightarrow$

$u_{00} = 0$, $u_{1,0} = 0.75$, $u_{2,0} = 1$, $u_{3,0} = 0.75$, $u_{4,0} = 0$

$u_{0,1} = 0$, $u_{4,1} = 0$

Auxilliary parameter: $z = \dfrac{k}{h^2} = 2$

Crank-Nikolson formula for $z = 2$

$(*)$ $6u_{i,j+1} - 2(u_{i+1,j+1} + u_{i-1,j+1}) = -2u_{ij} + 2(u_{i+1,j} + u_{i-1,j})$

In our case $j = 0$.

$\underline{\underline{i=1}}$: $6u_{1,1} - 2(u_{21} + u_{0,1}) = -2u_{10} + 2(u_{20} + u_{00})$

$\Rightarrow 6u_{11} - 2u_{21} = -1.5 + 4 = 2.5$

$\underline{\underline{i=2}}$: $6u_{2,1} - 2(u_{3,1} + u_{1,1}) = -2u_{20} + 2(u_{3,0} + u_{1,0})$

$\Rightarrow 6u_{21} - 2(u_{3,1} + u_{11}) = -2 + 3.5 = 1.5$

$\underline{\underline{i=3}}$: $6u_{3,1} - 2(u_{21} + u_{41}) = -2u_{30} + 2(u_{20} + u_{4,0})$

$\Rightarrow 6u_{31} - 2u_{21} = -1.5 + 4 = 2.5$

Write all this together:

$$6u_{11} - 2u_{21} = 2.5$$
$$-2u_{11} + 6u_{21} - 2u_{31} = 1.5$$
$$-2u_{21} + 6u_{31} = 2.5$$

Comment: By my mistake, when copying the Crank-Nikolson formula to the formula page, I copied the formula which corresponds to $z = 1$. Those who applied this formula correctly will receive the full credit.