Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4130, Fall 2007, Solutions

Problem 1 Find function $y(t),\ t \geq 0$ such that $y(0) = 0$ and

$$\int_0^t y'(u)y(t-u)du = t^2, \quad t > 0. \tag{1}$$

Solution. Let $Y(s) = \mathcal{L}y(s)$. Since $y(0) = 0$ we obtain $\mathcal{L}y'(s) = sY(s)$. The left-hand side of the equation (1) is the convolution of y and y'. Applying the Laplace transform to the both sides one obtains

$$sY^2(s) = \frac{2}{s^3}.$$

Therefore

$$Y(s) = \pm\frac{\sqrt{2}}{s^2}$$

and

$$\underline{y(t) = \mathcal{L}^{-1}Y(t) = \pm\sqrt{2}\ t.}$$

Problem 2

a. Let a function $f(x)$, $x \in (0, 2\pi)$ be defined by the relation

$$f(x) = \begin{cases} x, & \text{if } 0 \leq x \leq \pi; \\ 2\pi - x, & \text{if } \pi \leq x \leq 2\pi. \end{cases}$$

Find the sine Fourier series of f.

Solution. The general formula has the form

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\ dx.$$

In the our case $L = 2\pi$ and

$$b_n = \frac{1}{\pi}\int_0^{\pi} x\sin\frac{n}{2}x\ dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(2\pi - x)\sin\frac{n}{2}x\ dx = I_1 + I_2.$$

A direct evaluation of the integrals gives

$$I_1 = -\frac{1}{\pi}\cos\frac{n\pi}{2} + \frac{4}{\pi n^2}\sin\frac{n\pi}{2}, \quad I_2 = \frac{1}{\pi}\cos\frac{n\pi}{2} + \frac{4}{\pi n^2}\sin\frac{n\pi}{2},$$

and

$$b_n = I_1 + I_2 = \frac{8}{\pi n^2}\sin\frac{n\pi}{2} = \begin{cases} 0, & \text{if } n = 2k; \\ (-1)^k\frac{8}{\pi(2k+1)^2}, & \text{if } n = 2k+1. \end{cases}$$

Finally

$$\underline{f(x) \sim \frac{8}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \sin \frac{2k+1}{2} x} \tag{2}$$

b. Find all solutions of the form $u(x,t) = X(x)T(t)$ for the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial t^2} + u, \quad 0 < x < 2\pi,\ t > 0, \tag{3}$$

which satisfy the boundary conditions

$$u(0,t) = 0,\ u(2\pi,t) = 0, \quad t > 0. \tag{4}$$

Solution. Substituting a function u of the form $u(x,t) = X(x)T(t)$ we obtain

$$\frac{T'(t)}{T(t)} - 1 = \frac{X''(x)}{X(x)} = k.$$

This is a constant because it is independent both of t and x. Taking into account the boundary conditions (4) we obtain an equation with respect to X:

$$X''(x) - kX(x) = 0,\ 0 > x < 2\pi, \quad \text{and } X(0) = 0,\ X(2\pi) = 0. \tag{5}$$

and an equation with respect to T:

$$T'(t) - (k+1)T(t) = 0,\ t > 0. \tag{6}$$

The standard analysis shows that the only possible k's for which the problem (5) has non-trivial solutions are of the form

$$k_n = -\left(\frac{n}{2}\right)^2,\ n = 1, 2, \ldots.$$

The corresponding solutions of (5) are of the form

$$X_n(x) = A_n \sin \frac{n}{2} x,\ n = 1, 2, \ldots\ .$$

For each k_n equation (6) takes now the form

$$T_n'(t) - \left(1 - (\frac{n}{2})^2\right) T_n(t) = 0.$$

These equations have solutions

$$T_n(t) = C_n e^{1-(\frac{n}{2})^2 t}.$$

Finally we obtain

$$\underline{u_n(x,t) = B_n e^{1-(\frac{n}{2})^2 t} \sin \frac{n}{2} x},\ n = 1, 2, \ldots\ .$$

c. Find the solution of the problem (3), (4) formulated in section **b**, which in addition satisfies the initial condition

$$u(x,0) = f(x), \; 0 < x < 2\pi,$$

here the function f is defined in section **a.**

Answer:

$$\underline{u(x,t) = \frac{8}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} e^{-(k^2+k-3/4)t} \sin\frac{2k+1}{2}x}$$

Problem 3 Given the function

$$f(x) = \begin{cases} \cos x, & \text{if} \quad |x| \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

Find the Fourier transform of f and evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\sin 2w}{w} \cos w \; dw.$$

Solution. We use

$$\cos x = \frac{1}{2}\Big(e^{ix} + e^{-ix}\Big).$$

Therefore

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\frac{1}{2}\int_{-1}^{1} e^{-ix(w-1)}dx + \frac{1}{\sqrt{2\pi}}\frac{1}{2}\int_{-1}^{1} e^{-ix(w+1)}dx.$$

A direct calculation gives

$$\frac{1}{\sqrt{2\pi}}\frac{1}{2}\int_{-1}^{1} e^{-ix(w-1)}dx = \frac{1}{\sqrt{2\pi}}\frac{\sin(w-1)}{w-1} \tag{7}$$

$$\frac{1}{\sqrt{2\pi}}\frac{1}{2}\int_{-1}^{1} e^{-ix(w+1)}dx = \frac{1}{\sqrt{2\pi}}\frac{\sin(w+1)}{w+1} \tag{8}$$

Therefore

$$\underline{\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\Big(\frac{\sin(w-1)}{w-1} + \frac{\sin(w+1)}{w+1}\Big).}$$

In order to evaluate the integral put $u = w - 1$ in (7) (equivalently you may put $u = w + 1$ in (8)). You will obtain

$$\frac{1}{\sqrt{2\pi}}\frac{1}{2}\int_{-1}^{1} e^{-ixu}dx = \frac{1}{\sqrt{2\pi}}\frac{\sin u}{u}.$$

Inverse Fourier transform formula yields

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin u}{u} e^{iux}du = \begin{cases} \frac{1}{2}, & \text{if } |x| < 1; \\ \frac{1}{4}, & \text{if } |x| = 1; \\ 0, & \text{if } |x| > 1. \end{cases}$$

In this formula change variables $u = 2w$ and put $x = 1/2$:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin 2w}{w}e^{iw}dw = \frac{1}{2}.$$

Take the real part of the both sides and multiply by 2π:

$$\underline{\int_{-\infty}^{\infty}\frac{\sin 2w}{w}\cos w\ dw = \pi.}$$

Problem 4 You are given the problem

$$\begin{aligned} u''' + (3-u')u'' + 2u &= 0 \\ u(0) &= 1 \\ u'(0) &= 2 \\ u''(0) &= 5. \end{aligned} \tag{9}$$

a) Write the problem as a system of equations.
We introduce

$$\begin{aligned} u_1 &= u \\ u_2 &= u' \\ u_3 &= u'' \end{aligned}$$

which yields the system

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= u_3 \\ u_3' &= -\left((3-u_2)\,u_3 + 2u_1\right) \end{aligned}$$

with initial conditions

$$u_1(0) = 1 \qquad u_2(0) = 2 \qquad u_3(0) = 5.$$

Heun's method can be viewed as a predictor-corrector combination of Euler's method and the trapezoidal rule.

Backward Euler is given by

$$\mathbf{u}_{n+1} = \mathbf{u}_n + hf\left(t_{n+1}, \mathbf{u}_{n+1}\right).$$

Give a method using Euler's method as a predictor and backward Euler as a corrector.

The method can be written as

$$\begin{aligned} u_{n+1}^* &= u_n + hf(t_n, u_n) \\ u_{n+1} &= u_n + hf\left(t_{n+1}, u_{n+1}^*\right). \end{aligned}$$

b) Apply one step of the method you obtained in **a)** to (9). Use $h = 0.1$.

$$\mathbf{u}_1^* = \begin{bmatrix}1\\2\\5\end{bmatrix} + h\begin{bmatrix}2\\5\\-\left((3-2)\,5+2\cdot 1\right)\end{bmatrix} = \begin{bmatrix}1.20\\2.50\\4.30\end{bmatrix}$$

$$\mathbf{u}_1 = \begin{bmatrix}1\\2\\5\end{bmatrix} + h\begin{bmatrix}2.50\\4.30\\-\left((3-2.50)\cdot 4.30+2\cdot 1.20\right)\end{bmatrix} = \begin{bmatrix}1.25\\2.43\\5.03\end{bmatrix}$$

If you did not manage to find the method in **a)**, use Heun's method instead. [1]

$$\mathbf{k}_1 = \begin{bmatrix}2\\5\\-\left((3-2)\,5+2\cdot 1\right)\end{bmatrix} = \begin{bmatrix}2\\5\\-7\end{bmatrix}$$

$$\mathbf{k}_2 = \mathbf{f}\left(h, \mathbf{u}_0 + h\mathbf{k}_1\right) = \begin{bmatrix}2.50\\4.30\\-4.55\end{bmatrix}$$

$$\mathbf{u}_1 = \mathbf{u}_0 + \frac{h}{2}\left(\mathbf{k}_1 + \mathbf{k}_2\right) = \begin{bmatrix}1.23\\2.47\\4.42\end{bmatrix}$$

Problem 5

a) Find the polynomial $p_2(x)$ which interpolates

x_k	-2	-1	2
f_k	-13	-5	7

using Lagrangian interpolation.

We need the cardinal polynomials, they are given by

$$\begin{aligned}
l_0(x) &= \frac{(x+1)(x-2)}{(-2+1)(-2-2)} &&= \frac{1}{4}(x+1)(x-2)\\
l_1(x) &= \frac{(x+2)(x-2)}{(-1+2)(-1-2)} &&= -\frac{1}{3}(x+2)(x-2)\\
l_2(x) &= \frac{(x+2)(x-1)}{(2+2)(2+1)} &&= \frac{1}{12}(x+2)(x+1).
\end{aligned}$$

We now find our polynomial as

$$p_2(x) = f_0 l_0(x) + f_1 l_1(x) + f_2 l_2(x) = -x^2 + 5x + 1.$$

b) We then add another datapoint. We now want to find the polynomial $p_3(x)$ of the lowest

x_k	-2	-1	2	3
f_k	-13	-5	7	4

.

You can choose how you find this polynomial yourself.

[1] There was an error in the formula list. This will be taken into consideration during the grading.

In order to save ourself some work here, we can use the fact that a polynomial is uniquely given by its data and nodal values. This means that we can reuse $p_2(x)$. We use Newtonian interpolation to find the polynomial as

$$p_3(x) = p_2(x) + f[x_0, x_1, x_2, x_3]\,(x - x_0)\,(x - x_1)\,(x - x_2)\,.$$

This yields

x_k	f_k			
-2	-13	8	-1	-3/20
-1	-5	4	-7/4	
2	7	-3		
3	4.			

We now insert this to get

$$p_3(x) = p_2(x) - \frac{3}{20}\,(x + 2)\,(x + 1)\,(x - 2) = -\frac{3}{20}x^3 - \frac{23}{20}x^2 + \frac{28}{5}x + \frac{8}{5}.$$