1a Find inverse Laplace transform:

$$X(s) = \frac{s}{(s-1)^2+1} \; ; \quad Y(s) = \frac{s-2}{(s-1)^2+1}$$

$$X(s) = \frac{s-1}{(s-1)^2+1} + \frac{1}{(s-1)^2+1}$$

$$Y(s) = \frac{s-1}{(s-1)^2+1} - \frac{1}{(s-1)^2+1}$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s-1)^2+1}\right) = e^t \sin t \; ; \quad \mathcal{L}^{-1}\left(\frac{s-1}{(s-1)^2+1}\right) = e^t \cos t$$

$$\mathcal{L}^{-1}(X) = e^t(\cos t + \sin t) \; ;$$

$$\mathcal{L}^{-1}(Y) = e^t(\cos t - \sin t) \, .$$

1b Solve the initial value problem:

$$\begin{cases} x'(t) = x(t) + y(t) \\ y'(t) = -x(t) + y(t), \end{cases} \quad t > 0; \quad x(0) = 1, \quad y(0) = 1$$

$$\begin{cases} sX(s) - 1 = X(s) + Y(s) \\ sY(s) - 1 = -X(s) + Y(s) \end{cases} \Rightarrow \begin{cases} (s-1)X(s) - Y(s) = 1 \\ X(s) + (s-1)Y(s) = 1 \end{cases}$$

$$\left.\begin{aligned} (s-1)^2 X(s) - (s-1)Y(s) &= s-1 \\ X(s) + (s-1)Y(s) &= 1 \end{aligned}\right\} \Rightarrow \left[(s-1)^2 + 1\right] X(s) = s$$

$$\Rightarrow X(s) = \frac{s}{(s-1)^2 + 1}$$

$$\left.\begin{aligned} (s-1)X(s) - Y(s) &= 1 \\ (s-1)X(s) + (s-1)^2 Y(s) &= s-1 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \left[(s-1)^2 + 1\right] Y(s) = s-2 \Rightarrow Y(s) = \frac{s-2}{(s-1)^2 + 1}$$

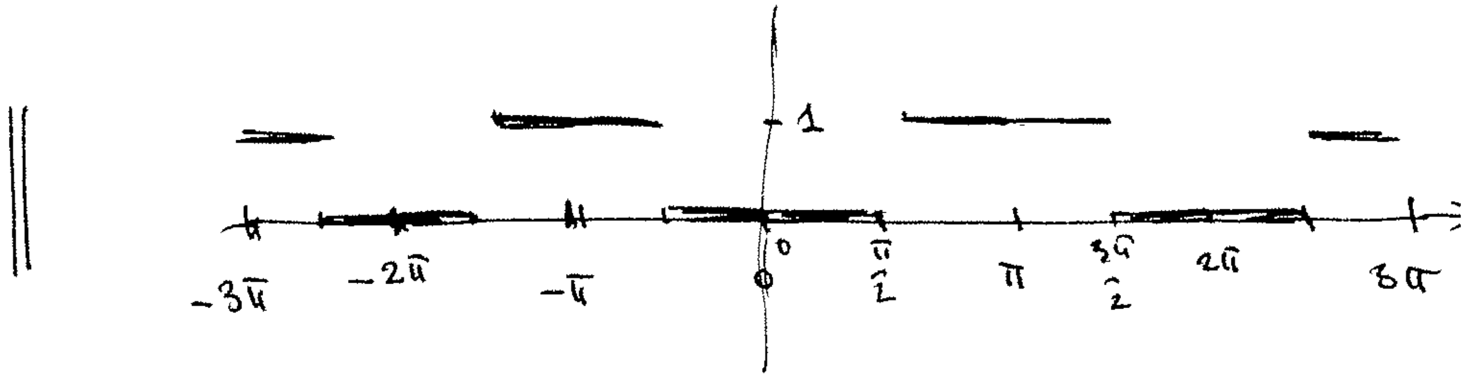
From the previous question:

$$x(t) = e^t(\cos t + \sin t)$$

$$y(t) = e^t(\cos t - \sin t)$$

2a $\quad f(x) = \begin{cases} 0 & 0 < x < \pi/2 \\ 1 & \pi/2 < x < \pi \end{cases}$

Even 2π-periodic prolongation:



Fourier series:

- Since f is even all sin-coeff. vanish:

$$f(x) \sim a_0 + \sum_1^\infty a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^\pi f(x)\,dx = \frac{1}{2}$$

$$n > 0 \Rightarrow \quad a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx = \frac{2}{\pi}\int_{\pi/2}^{\pi} \cos nx\,dx =$$

$$= \frac{2}{\pi}\left(\frac{\sin nx}{n}\right)\Bigg|_{x=\pi/2}^{\pi} = -\frac{2}{n\pi}\sin\frac{n\pi}{2} = \begin{cases} 0 & n\text{-even} \\ \frac{2}{n\pi}(-1)^{k+1} & \text{if } n = 2k+1 \end{cases}$$

$$\Rightarrow f(x) \sim \frac{1}{2} - \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\cos(2k+1)x$$

2b Find all functions of the form

$u(x,t) = X(x)\,T(t)$

which satisfy the equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t} - u$, $t > 0$, $0 < x < \pi$

and also $u_x(0,t) = 0$; $u_x(\pi,t) = 0$

$$X''(x)\,T(t) = X(x)\,\dot{T}(t) - X(t)\,T(t) \Rightarrow$$

$$\frac{X''(x)}{X(x)} = \frac{\dot{T}(t)}{T(t)} - 1 = k \text{ - constant.}$$

$$\Rightarrow \cancel{X'' - kX} \quad X''(x) - kX(x) = 0, \quad 0 < x < \pi$$

$$\dot{T}(t) - (k+1)\,T(t) = 0, \quad t > 0$$

Boundary conditions $\Rightarrow$

$$X''(x) - kX(x) = 0, \quad X'(0) = 0, \quad X'(\pi) = 0$$

Standard analysis:

$k = -p^2$, $p = 0, 1, 2, \ldots$

$$\cancel{p = 0 \Rightarrow X_0(x) = C_0}$$

$$\cancel{p \geq 0 \Rightarrow X_p(t) = C_p \cos pt}$$

$p = 0 \Rightarrow X_0(x) = a_0$; $p > 0 \Rightarrow X_p(x) = a_p \cos px$

2b (Continuation)

$$p=0 \Rightarrow \dot{T}_0(t) - T(t) = 0 \Rightarrow T(t) = e^{t} \cdot \text{Const}$$

$$p=1 \Rightarrow \dot{T}_1(t) = 0 \Rightarrow T_1(t) = \text{Const}$$

$$p \geq 1 \Rightarrow \dot{T}_p(t) + (p^2-1) T_p(t) = 0 \Rightarrow \underbrace{T_p(t) = e^{-(p^2-1)t}}\cdot \text{Con}$$

This formula can be used for all p's. I wrote the cases p = 0 and p = 1 separately, because in these cases the solution does not decay as $t \to +\infty$.

Finally:

$$u_0(x,t) = a_0 e^{t}$$
$$u_1(x,t) = a_1 \cos x$$
$$u_p(x,t) = a_p e^{-(p^2-1)t} \cos px$$

2c • Find the function $u(x,t)$ which meets the above equation and also $u(x,0) = f(x)$, $0<x<\pi$.

$$u(x,t) = \frac{1}{2} e^{t} - \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} e^{-k(k+2)t} \cos(2k+1)x$$

• Find the function $u(x,t)$ which meets the above equation and also $u(x,0) = \cos x \cos^2 x$, $0<x<\pi$.

I do not remember if there is a formula for $\cos^3 x$ in Rotman, but

$$\cos^2 x = \frac{1}{2}(1+\cos 2x) \Rightarrow$$

$$\cos x \cos^2 x = \frac{1}{2}\cos x + \frac{1}{2}\cos x \cos 2x =$$

$$= \frac{1}{2}\cos x + \frac{1}{2}\left[\frac{1}{2}\left(\cos(2x-x) + \cos(2x+x)\right)\right] =$$

$$= \frac{1}{4}\cos 3x + \frac{3}{4}\cos x$$

$$\Rightarrow u(x,t) = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x \cdot e^{-8t}$$

<u>Comment</u>: You may wanted to calculate the Fourier coefficients by the standard formulas, it will lead you to the same result but takes more time.

P 3 Find $f(t)$, $t>0$ such that

$$\int_0^t f(t-\tau)\, e^{3\tau}\, d\tau = \sin t \,, \quad t>0$$

Convolution Thm

$F(s) := (\mathcal{L} f)(s)$

Laplace transform

$$\mathcal{L}(e^{3t})(s) = \frac{1}{s-3}$$

$$\mathcal{L}(\sin t)(s) = \frac{1}{s^2+1}$$

$$\Rightarrow F(s)\,\frac{1}{s-3} = \frac{1}{s^2+1} \Rightarrow$$

$$\Rightarrow F(s) = \frac{s}{s^2+1} - \frac{3}{s^2+1} \Rightarrow \underline{\underline{f(t) = \cos t - 3\sin t}}$$

& P4 Evaluate convolution

$$(u(t+1)-u(t-1)) * e^{-|t|}$$

and find its Fourier transform.

$$u(t+1)-u(t-1) = \begin{cases} 1, & |t|<1 \\ 0 & \text{otherwise} \end{cases}$$

~~Fourier transform:~~

$$\cancel{\int_{-1}^{1} e^{-2i\pi\omega t}\,dt = \frac{-1}{2i\pi\omega}\left(e^{-2i\pi\omega} - e^{2i\pi\omega}\right) = \sin 2\pi}$$

Fourier transform:

$$\frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i\omega t}\,dt = \frac{-1}{\sqrt{2\pi}} \frac{e^{-i\omega} - e^{i\omega}}{i\omega} = \sqrt{\frac{2}{\pi}} \frac{\sin\omega}{\omega}$$

$$\mathcal{F}(e^{-|t|}) = \underbrace{\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(1+i\omega)t}\,dt}_{J_1} + \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(1-i\omega)t}\,dt}_{J_2} =$$

$$J_1 = \frac{1}{\sqrt{2\pi}} \frac{+1}{1+i\omega}; \quad J_2 = \frac{1}{\sqrt{2\pi}} \frac{1}{1-i\omega}$$

$$J_1 + J_2 = \frac{1}{\sqrt{2\pi}} \frac{1-i\omega+1+i\omega}{1+\omega^2} = \sqrt{\frac{2}{\pi}} \frac{1}{1+\omega^2}$$

Fourier transform of the convolution

$$\sqrt{2\pi}\, \frac{2}{\pi} \frac{\sin\omega}{\omega} \frac{1}{1+\omega^2} =$$

$$= \frac{2\sqrt{2}}{\sqrt{\pi}} \frac{\sin\omega}{\omega(1+\omega^2)}$$

P4 (Continuation)

Denote $f(t) = [u(t+1) - u(t-1)] * e^{-|t|} =$

$$= \int_{-1}^{1} e^{-|t-\tau|} d\tau$$

- $t > 1 \Rightarrow |t-\tau| = t-\tau$ for all τ, $-1<\tau<1$

$$\Rightarrow f(t) = \int_{-1}^{1} e^{-(t-\tau)} d\tau = e^{-t} \int_{-1}^{1} e^{\tau} d\tau = (e - \tfrac{1}{e}) e^{-t}$$

- $t < -1 \Rightarrow |t-\tau| = -t+\tau$ for all τ, $-1<\tau<1$

$$\Rightarrow f(t) = \int_{-1}^{1} e^{t-\tau} d\tau = e^{t} \int_{-1}^{1} e^{-\tau} d\tau = (e - \tfrac{1}{e}) e^{t}$$

- $-1 < t < 1 \Rightarrow |t-\tau| = \begin{cases} t-\tau, & \tau < t \\ \tau - t, & \tau > t \end{cases}$

$$f(t) = \int_{-1}^{t} e^{-t+\tau} d\tau + \int_{t}^{1} e^{-\tau+t} d\tau$$

$$\int_{-1}^{t} e^{-t+\tau} d\tau = e^{-t} \int_{-1}^{t} e^{\tau} d\tau = e^{-t}\left(e^{t} - \tfrac{1}{e}\right) = 1 - e^{-t-1}$$

$$+ \quad \int_{t}^{1} e^{-\tau+t} d\tau = e^{t} \int_{t}^{1} e^{-\tau} d\tau = e^{t}\left(-\tfrac{1}{e} + e^{-t}\right) = -e^{t-1} + 1$$

$$f(t) = 2 - \tfrac{1}{e}(e^{t} + e^{-t})$$

Finally write all together:

$$f(t) = \begin{cases} (e - e^{-1}) e^{-t}, & t \geq 1 \\ 2 - e^{-1}(e^{t} + e^{-t}), & -1 < t < 1 \\ (e - e^{-1}) e^{t}, & t \leq 1 \end{cases}$$

PS a Find polynomial of smallest possible degree, which solves the interpolation problem:

t_n	-2	-1	0	1	2
$p(t_n)$	3	1	1	3	7

$$\underline{\underline{p(t) = 1 + t + t^2}}$$

PS b Apply the Simpson method for evaluating

$$\int_{-2}^{2} p(t)\, dt$$

with nodes at the points $-2, -1, 0, 1, 2$; and compare the answer with the precise value of the integral.

Simpson: $\frac{1}{3}(3 + 4 + 2 + 12 + 7) = \frac{28}{3}$

You need not evaluate the ~~precise~~ explicit value of the integral because the Simpson method is precise for polynomials of degree two.

alternatively you may write

$$\int_{-2}^{2} (1 + t + t^2)\, dt = 4 + \tfrac{1}{3}16 = \frac{28}{3}.$$

P6 Apply the Gauss-Seidel iteration (two steps) for the system

$$\begin{cases} 2x_1 - x_2 = 3 \\ -x_1 + x_2 - x_3 = 0 \\ -x_2 - x_3 = 3 \end{cases}$$

starting from the point $(0,0,0)^T$

Step 1

$$\begin{cases} 2x_1^{(1)} = 3 \\ -x_1^{(1)} + x_2^{(1)} = 0 \\ -x_2^{(1)} - x_3^{(1)} = 3 \end{cases} \Rightarrow \quad \begin{aligned} x_1^{(1)} &= 3/2 \\ x_2^{(1)} &= 3/2 \\ x_3^{(1)} &= -\frac{3}{2} - 3 = -\frac{9}{2} \end{aligned}$$

Step 2

$$2x_1^{(2)} - \frac{3}{2} = 3$$

$$\begin{cases} -x_1^{(2)} + x_2^{(2)} + \frac{9}{2} = 0 \\ -x_2^{(2)} - x_3^{(2)} = 3 \end{cases} \Rightarrow$$

$$\Rightarrow x_1^{(2)} = \frac{9}{4}, \quad x_2^{(2)} = -\frac{9}{4}$$

$$x_3^{(2)} = +\frac{9}{4} - 3 = -\frac{3}{4}$$

$$x_1^{(2)} = 9/4$$

$$x_2^{(2)} = -9/4$$

$$x_3^{(2)} = -3/4$$