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Some Notes on the Behaviour of the Inverse Laplace Transform

Consider the following problem:

Compute, if possible, the inverse Laplace transform of $\ln(s)$, $s > 0$.

Your first instinct might be to make use of the differentiation-of-transforms result from Section 6.6 of the textbook. Writing $F(s) = \ln(s)$ and denoting the inverse transform as f, the calculation would go as follows:

$$\begin{aligned} -tf(t) &= \mathscr{L}^{-1}(F')(t) = \mathscr{L}^{-1}(1/s)(t) = 1 \\ \Rightarrow \quad f(t) &= -1/t, \end{aligned}$$

for $t > 0$. Let us check if the above is correct. Naively, let us attempt to compute the Laplace transform of $1/t$. We use the definition, and break the defining integral into two:

$$\mathscr{L}(1/t)(s) = \int_0^\infty \frac{e^{-st}}{t} dt = \int_0^1 \frac{e^{-st}}{t} dt + \int_1^\infty \frac{e^{-st}}{t} dt.$$

Let $s > 0$. Note that the second integral is a finite number that depends on s; call it $M(s)$. Then

$$\begin{aligned} \mathscr{L}(1/t)(s) &\geq \int_0^1 \frac{e^{-s}}{t} dt + M(s) \qquad (s > 0) \\ &= e^{-s} \left[\lim_{a \to 0^+} \ln(t) \Big|_{t=a}^{1} \right] + M(s) \\ &= e^{-s} \lim_{a \to 0^+} \ln(1/a) + M(s) \qquad (1) \end{aligned}$$

One would expect to obtain $-\ln(s)$, but (1) shows that $\mathscr{L}(1/t)(s) = +\infty$ for every $s > 0$. **What could have gone wrong?**

The above contradiction demonstrates the importance of paying attention to the assumptions stated in the theorems we want to use. The result in Section 6.6 that gives us the formula $-tf(t) = \mathscr{L}^{-1}(F')(t)$ *is based on the assumption that* f *is piecewise continuous on* $[0, \infty)$ *and satisfies a certain growth restriction: namely, there exist constants* $M, k > 0$ *such that* $|f(t)| \leq Me^{kt}$ $\forall t \geq 0$.

Without the above conditions on f, F may not even be differentiable! The manner in which the above problem was "solved", *no checks were performed* to see if the conditions under which we may use the formula $-tf(t) = \mathscr{L}^{-1}(F')(t)$ are obeyed. The contradiction in the above calculation shows that the formula $-tf(t) = \mathscr{L}^{-1}(F')(t)$ cannot be used for $F(s) = \ln(s)$. This gives rise to the following natural

Question: *If we are given an $F(s)$, and are asked to determine its inverse Laplace transform — i.e., we know nothing about the continuity or growth of f — how do we know whether we can apply any of the theorems in Chapter 6 of the textbook ?*

It is easy to see that for any f that satisfies the aforementioned condition:

- $\mathscr{L}(f)(s)$ exists for every $s > k$,
- $|\mathscr{L}(f)(s)| \leq \dfrac{M}{s-k} \ \ \forall s > k,$ (2)

where M and k are as discussed above. (Can you show why this is so ?) Therefore, if you are given some $F(s)$ that does not satisfy (2) for any $M, k > 0$, then *you cannot apply many of the theorems in Chapter 6.* Now, inequalities are somewhat difficult to deal with in a course at the level of Matematikk 4N. Therefore, in the exam, you will have to deal with only those kinds of $F(s)$ for which all the theorems of Chapter 6 are applicable.

We can say a little more. Let $\mathscr{F}$ denote the collection of all those functions that are Laplace transforms of functions that are continuous on $[0, \infty)$ and satisfy a growth restriction of the above form. Suppose you are given any function F from the collection $\mathscr{F}$. Then, regardless of the theorem you use to find its inverse transform, $\mathscr{L}^{-1}(F)$ will be continuous on $[0, \infty)$. This follows from the comments on **uniqueness** given on page 210 of the 10th edition of Kreyszig (or on page 226 of the 9th edition of Kreyszig).

Example: The inverse Laplace transform f of the function $F(s) = \ln((1+s^2)/(s+2)^2)$ — which was given to you in an exercise in Task 3 — might look like it is undefined at $t = 0$. But if you examine the function f (this f will be **explicitly** stated here once the submission date of Task 3 has passed !) carefully, you will see that $\lim_{t\to 0^+} f(t) = 4$. Thus, the inverse transform of this F is continuous on $[0, \infty)$.