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Notes

Some Notes on Normal Forms for Second-order PDEs

This note is meant primarily to explain the notion of normal forms more clearly (in comparison to Section 12.4 of the textbook) and to correct a **major error** in the second table in Section 12.4. For greater clarity, we shall only discuss linear, *homogeneous*, second-order PDEs in two variables (which is the focus of interest in our course). Thus, we consider PDEs of the form (with the understanding that $A, B, C \not\equiv 0$)

$$A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} \;=\; 0. \tag{1}$$

The idea behind the concept "normal forms" is to copy the trick in D'Alembert's solution to the 1-dimensional wave equation to transform an equation of the form (1) to a simpler form. This means that we look for functions $y_1(x)$ and $y_2(x)$ (defined in the region where a solution to (1) is desired), and use these to define new independent variables

$$v \;=\; y - y_1(x), \qquad w \;=\; y - y_2(x),$$

so that equation (1) expressed in terms of v and w has a simpler form which we can hope to solve by hand. This simpler form is called the *normal form.* We obtained a *very* simple normal form for the 1-dimensional wave equation by taking $v = x + ct$ and $w = x - ct$. When $AC - B^2$ has a definite sign, we know exactly what the normal forms look like. These are summarised in the table below. Depending on whether $AC - B^2$ is negative, zero, or positive, we call the equation (1) *hyperbolic*, *parabolic*, or *elliptic*.

Type	New variables	Normal form in in general	Normal form when A, B, C are constants
Hyperbolic	$v = \Phi(x,y), \quad w = \Psi(x,y)$	$u_{vw} = F_1$	$u_{vw} = 0$
Parabolic	$v = x, \quad w = \Phi(x,y) = \Psi(x,y)$	$\boldsymbol{u_{vv} = F_2}$	$\boldsymbol{u_{vv} = 0}$
Elliptic	$v = \dfrac{(\Phi+\Psi)(x,y)}{2} \quad w = \dfrac{(\Phi-\Psi)(x,y)}{2i}$	$u_{vv} + u_{ww} = F_3$	$u_{vv} + u_{ww} = 0$

Here, F_1, F_2 and F_3 are certain expressions involving v, w, u_v and u_w. The two important things to notice are:

1) When A, B and C are constants, the normal forms are exceptionally simple.
2) The normal form for a parabolic equation **has been stated incorrectly** in the book. I have provided the correct normal form **in bold** in the table above.

I do not plan to explain how we conclude that the normal forms are as above. However, I shall justify that the normal form for the parabolic case, as given above, is correct. But first: how do we calculate $\Phi(x,y)$ and $\Psi(x,y)$. **Answer:** We first find the *two general solutions* to the ODE:

$$A(x,y)\left(\frac{dy}{dx}\right)^2 - 2B(x,y)\left(\frac{dy}{dx}\right) + C(x,y) \;=\; 0. \tag{2}$$

If we express these two solutions as $y = y_1(x) + C_1$ and $y = y_2(x) + C_2$ (where C_1 and C_2 are constants of integration), then take $\Phi(x,y) = y - y_1(x)$, $\Psi(x,y) = y - y_2(x)$.
The equation (2) is called the *characteristic equation*. This equation has two general solutions because, when we solve for dy/dx using the quadratic formula, we get two *different* ODEs:

$$\frac{dy}{dx} \;=\; \frac{B}{A}(x,y) + \frac{\sqrt{B^2(x,y) - A\cdot C(x,y)}}{A(x,y)} \quad\text{and}\quad \frac{dy}{dx} \;=\; \frac{B}{A}(x,y) - \frac{\sqrt{B^2(x,y) - A\cdot C(x,y)}}{A(x,y)},$$

except when (1) *is parabolic*, in which case both equations are the same. That is why, in the table above, it is written that $\Phi(x,y) = \Psi(x,y)$ for the parabolic case.
Those who are interested in checking that the normal form given above for the parabolic case is the correct form should read ahead. Note that $v = x$ and $w = y - Y(x)$, where $y = Y(x) + C$ is the single general solution of the characteristic equation. So, by the chain rule for partial derivatives:

$$\begin{aligned} u_x \;&=\; u_v v_x + u_w w_x \;=\; u_v - u_w\, Y'(x), \qquad u_y \;=\; u_v v_y + u_w w_y \;=\; u_w, \\ u_{xx} \;&=\; u_{vv} - u_{vw}\, Y'(x) - u_{wv}\, Y'(x) + u_{ww}\,(Y')^2(x) - u_w\, Y''(x), \\ u_{yy} \;&=\; u_{ww}, \qquad u_{xy} \;=\; u_{vw} - u_{ww}\, Y'(x). \end{aligned}$$

If we are interested in twice-continuously differentiable solutions only, then $u_{vw} = u_{wv}$. Substituting all of the above into (1) gives us:

$$\begin{aligned} A(x,y)\, u_{vv} \;+\; 2\big[B(x,y) - A(x,y)\, Y'\big]\, u_{vw} & \\ +\; \big[A(x,y)\,(Y')^2 - 2B(x,y)\, Y' + C(x,y)\big]\, u_{ww} \;&=\; u_w\, Y''. \end{aligned}$$

Since $y = Y(x) + C$ is the general solution to the characteristic equation, the coefficient for u_{ww} above is zero. As $B^2 - AC = 0$, Y actually satisfies the equation $Y' = (B/A)(x,y)$ (see the solutions for dy/dx above). Hence, $B(x,y) - A(x,y)Y'$ is also zero. This reduces the above equation to

$$u_{vv} \;=\; F_2,$$

where

$$F_2 \;=\; \begin{cases} 0, & \text{if } A, B, C \text{ are all constants,} \\ \dfrac{u_w\, Y''(v)}{A(v, w + Y(v))}, & \text{otherwise.} \end{cases}$$

This justifies the normal form for the parabolic case.