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Notes

Heun's Method for a System of First-order ODEs

The purpose of this note is to give an overview, and work out an example, of Heun's method for a *system* of first-order ODEs. **This topic is on the syllabus of the exam**, even though it is not covered in Chapter 21.3 of the textbook. (Those of you who have consulted the list of formulas on numerical methods, provided in the exams, would already know that Heun's method is on the syllabus.)

This presentation closely follows the lecture of November 9th, 2012.

The standard initial-value problem for a system of first-order ODEs in the unknowns $y_1, y_2, \ldots, y_m$ can be rewritten in the following form (in what follows, bold-face characters will denote vector quantities):

$$\begin{aligned} \boldsymbol{y'} &= \boldsymbol{f}(x, \boldsymbol{y}), \\ \boldsymbol{y}(x_0) &= \boldsymbol{y}_0. \end{aligned} \tag{1}$$

In the above presentation, the vector $\boldsymbol{y}$ simply denotes the m unknown functions, i.e. $\boldsymbol{y} = (y_1, y_2, \ldots, y_m)$. The vector-valued function $\boldsymbol{f}$ also has m components: $\boldsymbol{f} = (f_1, f_2, \ldots, f_m)$. Therefore, written out in components, the initial-value problem (1) looks like this:

$$\begin{aligned} y_1' &= f_1(x, y_1, \ldots, y_m), \\ y_2' &= f_2(x, y_1, \ldots, y_m), \\ &\vdots \\ y_m' &= f_m(x, y_1, \ldots, y_m), \\ y_1(x_0) = y_{1,0},\ \ y_2(x_0) &= y_{2,0},\ \ldots,\ \ y_m(x_0) = y_{m,0}. \end{aligned}$$

Heun's method with step size h gives us approximations $\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3, \ldots$ to the values $\boldsymbol{y}(x_1), \boldsymbol{y}(x_2), \boldsymbol{y}(x_3), \ldots$ of the unknown solution $\boldsymbol{y}$ at each $x_n = x_0 + nh$, $n = 1, 2, 3, \ldots$. The rule is similar to the one for a single ODE; the only difference is that the various quantities involved are vectors. The rule is as follows

$$\begin{aligned} \boldsymbol{k}_1 &= h\boldsymbol{f}(x_n, \boldsymbol{y}_n) \\ \boldsymbol{k}_2 &= h\boldsymbol{f}(x_n + h, \boldsymbol{y}_n + \boldsymbol{k}_1) \\ \boldsymbol{y}_{n+1} &= \boldsymbol{y}_n + \frac{1}{2}(\boldsymbol{k}_1 + \boldsymbol{k}_2). \end{aligned} \tag{2}$$

Let us look at an example to see this method in practice.

Example 1 (adapted from prob. 12, Chap. 21.3; either Ed. 9 or 10 of Kreyszig) *Using Heun's method, find approximate values of* $y(1.1)$ *and* $y(1.2)$ *for the solution of the* *initial value problem*

$$x\,y'' + y' + xy \;=\; 0,$$
$$y(1) \;=\; 0.76520,\;\; y'(1) \;=\; -0.44005.$$

Round all outputs to 5 places of decimal.

We first have to convert the 2nd-order ODE into a system in two unknowns. By the usual method discussed in Chapter 21.3, i.e. taking $y_1 = y$ and $y_2 = y'$, we obtain the system

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= -\frac{y_2}{x} - y_1, \\ \boldsymbol{y}_0 &= (0.76520, -0.44005). \end{aligned} \tag{3}$$

Let us express $\boldsymbol{y}_n$ in components as $\boldsymbol{y}_n = (y_{1,n}, y_{2,n})$. By the same scheme, the intermediate quantities $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ will be written as $\boldsymbol{k}_1 = (k_{1,1}, k_{2,1})$ and $\boldsymbol{k}_2 = (k_{1,2}, k_{2,2})$. So, in terms of components:

$$\begin{aligned} k_{1,1} &= hf_1(x_n, y_{1,n}, y_{2,n}) \;=\; hy_{2,n} \\ k_{2,1} &= hf_2(x_n, y_{1,n}, y_{2,n}) \;=\; -h\left(\frac{y_{2,n}}{x_n} + y_{1,n}\right). \end{aligned}$$

I leave it to the students to figure out the component-wise expressions for $k_{1,2}$ and $k_{2,2}$. With thise expressions, we build the following table (note that the entries appearing below are rounded to the 5th place of decimal):

n	x_n	$y_{1,n}$	$y_{2,n}$	$k_{1,1}$	$k_{2,1}$	$k_{1,2}$	$k_{2,2}$	$y_{1,n+1}$	$y_{2,n+1}$
0	1	0.76520	-0.44005	-0.04401	-0.03252	-0.04726	-0.02916	0.71957	-0.47089
1	1.1	0.71957	-0.47089	-0.04709	-0.02915	-0.05000	-0.02558	0.67102	

Remember that $y_1 = y$. Thus, to approximate $y(1.1)$ and $y(1.2)$, we just need to determine $y_{1,n},\; n = 1, 2$. Hence, we do not really need to compute the last entry of the above table to complete this task. Our conclusions are:

$$y(1.1) \;\approx\; 0.71957,$$
$$y(1.2) \;\approx\; 0.67102.$$