Norwegian University of Science and Technology
Departement of Mathematical Sciences

Solutions to exercise set 11

1 $f^n = \begin{bmatrix} x_n^2 + x_n y_n^3 - 9 \\ 3x_n^2 y_n - y_n^3 - 4 \end{bmatrix}$

$J^n = \begin{bmatrix} 2x_n + y_n^3 & 3x_n y_n^2 \\ 6x_n y_n & 3x_n^2 - 3y_n^2 \end{bmatrix}$

Vi har $J^n \begin{bmatrix} \Delta x^n \\ \Delta y^n \end{bmatrix} = -f^n$ og får

n	x^n	y^n	Δx^n	Δy^n	$\lVert \Delta \mathbf{x}^n \rVert_2$
0	1.2	2.5	0.0558	-0.5420	0.5449
1	1.2558	1.9580	0.0671	-0.1852	0.1970
2	1.3228	1.7728	0.0134	-0.0184	0.0227
3	1.3362	1.7544	–	–	$2.09 \cdot 10^{-4}$
4	1.3364	1.7542	–	–	$7.91 \cdot 10^{-9}$
5	1.3364	1.7542	–	–	$2.53 \cdot 10^{-16}$

Her er $x^n = x^{n-1} + \Delta y^{n-1}$ og $x^n = y^{n-1} + \Delta y^{n-1}$. Den siste kolonnen med $\lVert \Delta \mathbf{x}^n \rVert_2 = \sqrt{(x^n)^2 + (y^n)^2}$ er oppgitt for se på konvergensen til metoden. Vi ser at vi har kvadratisk konvergens slik som vi hadde for Newtons metode med 1 variabel.

2 Consider the initial value problem

$$y' = -y, \qquad x \in [0, 1], \qquad y(0) = 1.$$

a) The general solution of

$$y' = -y$$

is $y(x) = C\mathrm{e}^{-x}$. The initial condition $y(0) = 1$ gives $y(x) = \mathrm{e}^{-x}$. Then we get $y(1) = \mathrm{e}^{-1}$.

b) We recall Euler's method for the first order initial value problem $y = f(x, y)$, $y(x_0) = y_0$ as $y_{n+1} = y_n + hf(x_n, y_n), n \geq 0$ in which $x_n = x_0 + nh$. In this case we obtain

$$y_{n+1} = y_n - hy_n$$

starting from $x_0 = 0$ and $y_0 = 1$. The results of four steps using this method, as well as the given exact solution, is collected in Table 1. We here have an error of 0.05473191.

n	x_n	y_n	$y(x_n)$
0	0	1.000000000	1.000000000
1	0.25	0.750000000	0.778800783
2	0.5	0.562500000	0.60653066
3	0.75	0.421875000	0.472366553
4	1	0.316406250	0.367879441

Table 1: Solutions using Euler's method

c) We recall Heun's method for the first order initial value problem $y = f(x, y)$, $y(x_0) = y_0$ as $y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$ where $k_1 = hf(x_n, y_n)$, $k_2 = hf(x_n + h, y_n + k_1)$ and $x_n = x_0 + nh$. In this case we obtain

$$k_1 = -hy_n$$
$$k_2 = -h(y_n + k_1)$$

starting from $x_0 = 0$ and $y_0 = 1$. The results of two steps using this method along with the stage values k_1 and k_2, are collected in Table 2. We here have an error of 0.022745559.

n	x_n	k_1	k_2	y_n	$y(x_n)$
0	0	–	–	1.000000000	1.000000000
1	0.5	-0.500000000	-0.250000000	0.625000000	0.60653066
2	1	-0.31250000	-0.156250000	0.390625000	0.367879441

Table 2: Solutions using Heun's method.

d) We recall classical 4th order Runge-Kutta method for the first order initial value problem $y = f(x, y)$, $y(x_0) = y_0$ as $y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ where $k_1 = hf(x_n, y_n)$, $k_2 = hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$, $k_3 = hf(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$, $k_4 = hf(x_n + h, y_n + k_3)$ and $x_n = x_0 + nh$. In this case we obtain

$$k_1 = -hy_n$$
$$k_2 = -h(y_n + k_1/2)$$
$$k_3 = -h(y_n + k_2/2)$$
$$k_4 = -h(y_n + k_3)$$

starting from $x_0 = 0$ and $y_0 = 1$. The results of one step using this method along with the stage values k_1, k_2, k_3, k_4 are collected in Table 3. We here have an error of 0.007120559.

n	x_n	k_1	k_2	k_3	k_4	y_n	$y(x_n)$
0	0	–	–	–	–	1.000000000	1.000000000
1	1	-1	-0.5	-0.75	-0.25	0.375000000	0.367879441

Table 3: Solutions using RK4

3 Consider now the initial value problem

$$y' = \frac{y}{x} - \left(\frac{y}{x}\right)^2, \qquad x \in [1,2], \qquad y(1) = 1.$$

a) If

$$y(x) = \frac{x}{1 + \ln x}$$

then

$$y'(x) = \frac{1(1+\ln x) - x\left(0 + \frac{1}{x}\right)}{(1+\ln x)^2} = \frac{1 + \ln x - 1}{(1+\ln x)^2} = \frac{\ln x}{(1+\ln x)^2}.$$

We also have

$$\frac{y(x)}{x} - \left(\frac{y(x)}{x}\right)^2 = \frac{1}{1+\ln x} - \frac{1}{(1+\ln x)^2} = \frac{1(1+\ln x) - 1}{(1+\ln x)^2} = \frac{\ln x}{(1+\ln x)^2},$$

so we do in fact have

$$y(x) = \frac{x}{1+\ln x}.$$

b) We recall Euler's method for the first order initial value problem $y = f(x,y)$, $y(x_0) = y_0$ as $y_{n+1} = y_n + hf(x_n, y_n), n \geq 0$ in which $x_n = x_0 + nh$. In this case we obtain

$$y_{n+1} = y_n + h\left(\frac{y_n}{x_n} - \left(\frac{y_n}{x_n}\right)^2\right)$$

starting from $x_0 = 1$ and $y_0 = 1$. The results of four steps using this method, as well as the given exact solution, is collected in Table 4. We here have an error of 0.029461711.

n	x_n	y_n	$y(x_n)$
0	1	1.00000000	1.00000000
1	1.25	1.00000000	1.021956907
2	1.5	1.04000000	1.067262354
3	1.75	1.09315556	1.122071226
4	2	1.151770507	1.181232218

Table 4: Solutions using Euler's method

c) We recall Heun's method for the first order initial value problem $y = f(x,y)$, $y(x_0) = y_0$ as $y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$ where $k_1 = hf(x_n, y_n)$, $k_2 = hf(x_n + h, y_n + k_1)$ and $x_n = x_0 + nh$. In this case we obtain

$$k_1 = h\left(\frac{y_n}{x_n} - \left(\frac{y_n}{x_n}\right)^2\right)$$

$$k_2 = h\left(\frac{y_n + k_1}{x_n + h} - \left(\frac{y_n + k_1}{x_n + h}\right)^2\right)$$

starting from $x_0 = 1$ and $y_0 = 1$. The results of two steps using this method along with the stage values k_1 and k_2, are collected in Table 5. We here have an error of 0.012646623.

n	x_n	k_1	k_2	y_n	$y(x_n)$
0	1	–	–	1.000000000	1.000000000
1	1.5	0.000000000	0.111111111	1.055555556	1.067262354
2	2	0.104252401	0.121807677	1.168585595	1.181232218

Table 5: Solutions using Heun's method

d) We recall classical 4th order Runge-Kutta method for the first order initial value problem $y = f(x, y)$, $y(x_0) = y_0$ as $y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ where $k_1 = hf(x_n, y_n)$, $k_2 = hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$, $k_3 = hf(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$, $k_4 = hf(x_n + h, y_n + k_3)$ and $x_n = x_0 + nh$. In this case we obtain

$$k_1 = h\left(\frac{y_n}{x_n} - \left(\frac{y_n}{x_n}\right)^2\right)$$
$$k_2 = h\left(\frac{y_n + k_1/2}{x_n + h/2} - \left(\frac{y_n + k_1/2}{x_n + h/2}\right)^2\right)$$
$$k_3 = h\left(\frac{y_n + k_2/2}{x_n + h/2} - \left(\frac{y_n + k_2/2}{x_n + h/2}\right)^2\right)$$
$$k_4 = h\left(\frac{y_n + k_3}{x_n + h} - \left(\frac{y_n + k_3}{x_n + h}\right)^2\right)$$

starting from $x_0 = 1$ and $y_0 = 1$. The results of one step using this method along with the stage values k_1, k_2, k_3, k_4 are collected in Table 6. We here have an error of 0.003013548.

n	x_n	k_1	k_2	k_3	k_4	y_n	$y(x_n)$
0	1	–	–	–	–	1.000000000	1.000000000
1	2	0	0.222222222	0.192043896	0.240779786	1.17821867	1.181232218

Table 6: Solutions using RK4

4 **a)**

$$y_1' = y_2$$
$$y_2' = \cos y_1$$

b)

$$y_{1,n+1} = y_{1,n} + hy_{2,n}$$
$$y_{2,n+1} = y_{2,n} + h\cos y_{1,n}$$

c) Sett $y = y_1$, det gir $y' = y_2$, $y'' = y_3$ og diffligningen blir

$$y''' = \cos y + \sin y' - \mathrm{e}^{y''} + x^2$$

5 **a)** Funksjonen er like. Da får vi at

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{1}{2}\pi,$$

og

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx = \frac{2}{\pi}(0 - \int_0^{\pi}\frac{\sin(nx)}{n}\,dx)) = \frac{2}{\pi}\frac{\cos(n\pi)-1}{n^2}$$

ved delvis integrasjon. Man ser at a_n er 0 for n like og $-\frac{4}{\pi n^2}$ for n odde. Rekka er gitt ved

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(nx).$$

b) Bruker Parseval. Man har at høyre side er

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2}{3}\pi^2.$$

Venstre side er

$$\frac{1}{2}\pi^2 + \sum_{k=0}^{\infty}\frac{16}{\pi^2(2k+1)^4}.$$

Ved å flytte over og gange opp får man

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4} = \frac{\pi^4}{96}.$$