Solutions to the Exam in Math 4N & 4M

Problem 1 (Only for Math 4N)

a) The solution requires the t-shifting theorem. We see from the table of Laplace transforms given that $1/(s+1)(s-1) = \mathscr{L}[\sinh t](s)$. Therefore, the desired inverse transform is

$$u(t-a)\sinh(t-a) \qquad \Big(\text{ or } u(t-a)\frac{e^{t-a} - e^{-(t-a)}}{2}\Big).$$

b) Applying the Laplace transform to the given ODE, and using the initial conditions, we get

$$(s^2Y(s) - s - 1) - Y(s) \;=\; 2e^{-s},$$

where Y denotes the Laplace transform of the solution y. This gives:

$$Y(s) \;=\; \frac{2e^{-s}}{s^2-1} + \frac{1}{s-1}.$$

Taking the inverse transform, and using the result of part (a), gives the solution $y(t) = 2u(t-1)\sinh(t-1) + e^t$.

Problem 2 (Problem 1 for Math 4M)

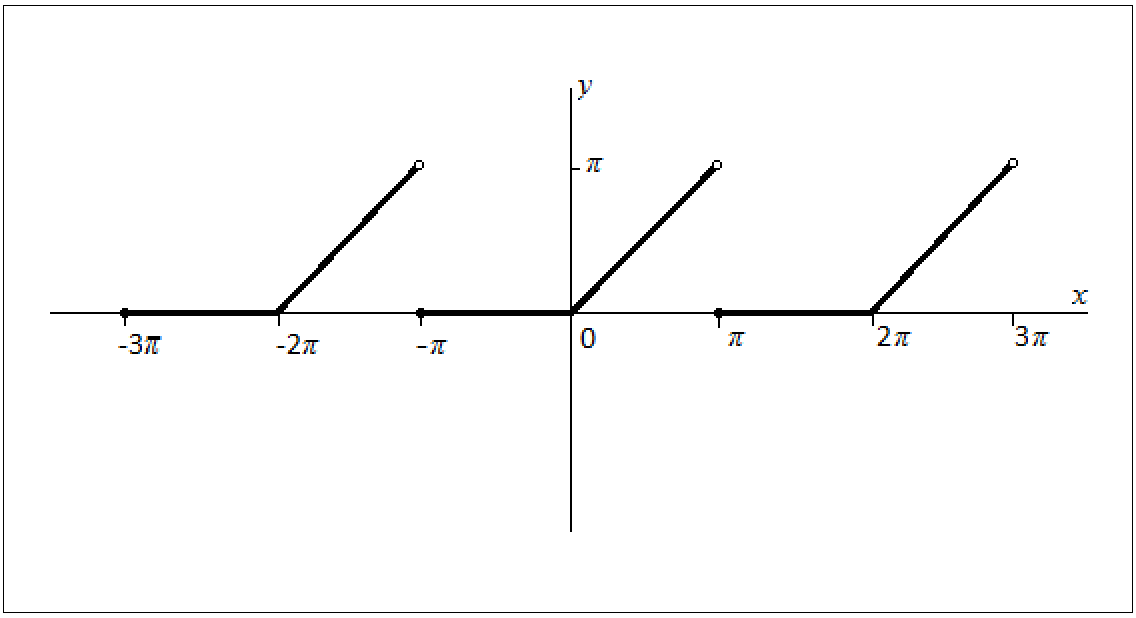
a) A sketch of the graph of the given f on the interval $[-3\pi, 3\pi)$ is given at the top of the next page.

The Fourier coefficient

$$A_0 \;=\; \frac{1}{2\pi}\int_0^\pi x\,dx \;=\; \pi/4.$$

The Fourier coefficients

$$\begin{aligned} A_n \;=\; \frac{1}{\pi}\int_0^\pi x\cos(nx)\,dx &= \frac{1}{\pi}\left[\frac{x\sin(nx)}{n}\bigg|_{x=0}^{\pi} - \frac{1}{n}\int_0^\pi \sin(nx)\,dx\right] \\ &= \frac{\cos(nx)}{\pi n^2}\bigg|_{x=0}^{\pi} \\ &= \frac{(-1)^n - 1}{\pi n^2}, \end{aligned}$$



for $n = 1, 2, 3, \dots$ The Fourier coefficients

$$B_n = \frac{1}{\pi}\int_0^\pi x\sin(nx)\,dx = \frac{1}{\pi}\left[-\frac{x\cos(nx)}{n}\Big|_{x=0}^{\pi} + \frac{1}{n}\int_0^\pi \cos(nx)\,dx\right]$$
$$= \frac{(-1)^{n+1}}{n} + \frac{\sin(nx)}{\pi n^2}\Big|_{x=0}^{\pi}$$
$$= \frac{(-1)^{n+1}}{n},$$

for $n = 1, 2, 3, \dots$ Therefore, the Fourier series of f is

$$\frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{\pi n^2}\cos(nx) + \frac{(-1)^{n+1}}{n}\sin(nx)\right]. \tag{1}$$

b) We can substitute $x = 0$ in (1). This procedure is *completely straightforward.*

We could also compute the given series by taking $x = \pi$ in (1). However, we need to be **careful** with this option because f has a jump-discontinuity at $x = \pi$. This implies that

$$\frac{f(\pi+) + f(\pi-)}{2} = \frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{(-1)^n - 1}{\pi n^2}\cos(n\pi).$$

Observe that $f(\pi-) = \pi$ and $f(\pi+) = 0$. Rearranging terms in the above equation, we get

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

Problem 3 (Problem 2 for Math 4M) Since $|x| = -x$ when $x < 0$ and $|x| = x$ otherwise, we have

$$\widehat{f}(w) = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} xe^{(a-iw)x}\,dx + \int_{0}^{\infty} xe^{-(a+iw)x}\,dx\right] \equiv \frac{1}{\sqrt{2\pi}}(I_1 + I_2).$$

Using integration by parts, we get:

$$\begin{aligned} I_1 &= \left.\frac{xe^{(a-iw)x}}{a-iw}\right|_{x=-\infty}^{0} - \frac{1}{a-iw}\int_{-\infty}^{0} e^{(a-iw)x}\,dx \\ &= \left.\frac{xe^{(a-iw)x}}{a-iw}\right|_{x=-\infty}^{0} - \left.\frac{e^{(a-iw)x}}{(a-iw)^2}\right|_{x=-\infty}^{0} \\ &= \lim_{t\to\infty}\left[-\frac{(-t)e^{-at}e^{iwt}}{a-iw} - \frac{1}{(a-iw)^2} + \frac{e^{-at}e^{iwt}}{(a-iw)^2}\right]. \end{aligned}$$

Since it is given that $a > 0$, and e^{iwt} is bounded for all t, the third term above converges to 0. Furthermore, we know that $t^p e^{-at} \to 0$, for any power p, as $t \to \infty$. Hence, the first term above converges to 0. Thus $I_1 = -1/(a-iw)^2$.

In a similar way, the calculation of I_2 proceeds as follows:

$$\begin{aligned} I_2 &= -\left.\frac{xe^{-(a+iw)x}}{a+iw}\right|_{x=0}^{\infty} + \frac{1}{a+iw}\int_{0}^{\infty} e^{-(a+iw)x}\,dx \\ &= -\left.\frac{xe^{-(a+iw)x}}{a+iw}\right|_{x=0}^{\infty} - \left.\frac{e^{-(a+iw)x}}{(a+iw)^2}\right|_{x=0}^{\infty} \\ &= \lim_{t\to\infty}\left[-\frac{te^{-at}e^{-iwt}}{a+iw} + \frac{1}{(a+iw)^2} - \frac{e^{-at}e^{-iwt}}{(a+iw)^2}\right]. \end{aligned}$$

By exactly the same arguments as above to justify the limits, we get $I_2 = 1/(a+iw)^2$. Combining the values of I_1 and I_2, we get

$$\begin{aligned} \widehat{f}(w) &= \frac{1}{\sqrt{2\pi}}\left[-\frac{1}{(a-iw)^2} + \frac{1}{(a+iw)^2}\right] \\ &= -\frac{4iaw}{\sqrt{2\pi}(a^2+w^2)^2}. \end{aligned} \tag{2}$$

To evaluate the given integral, we use the Fourier-inversion formula. By (2), have:

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-4iaw}{(a^2+w^2)^2}e^{iwx}\,dw \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-4iaw(\cos(wx) + i\sin(wx))}{(a^2+w^2)^2}\,dw. \end{aligned}$$

Since $f(x)$ is real-valued, the imaginary part on the right-hand side of the above equation must equal zero. Hence

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4aw\sin(wx)}{(a^2+w^2)^2}\,dw \quad \forall x \in \mathbb{R}.$$

The above holds true for each $x \in \mathbb{R}$ because f is a continuous function and, as $a > 0$, satisfies all the conditions of the theorem that gives us the Fourier-inversion formula. Taking $x = 1$ and $a = 1$ in the last equations gives us the answer:

$$\int_{-\infty}^{\infty} \frac{w(\sin(w))}{(1+w^2)^2}\,dw \;=\; \frac{\pi}{2}\,e^{-|x|}\Big|_{x=1} \;=\; \frac{\pi e^{-1}}{2}.$$

Problem 4 (Problem 3 for Math 4M)

a) Assuming u to be of the form $u(x,t) = F(x)G(t)$, we get

$$u_t \;=\; F\dot{G}, \qquad u_{xx} = F''G.$$

Substituting into the given PDE, we have $F\dot{G} = (F'' - 5F)G$. Since we seek non-trivial solutions, FG cannot be identically zero. Therefore, we divide the last equation by FG to get

$$\frac{\dot{G}}{G}(t) + 5 \;=\; \frac{F''(x)}{F(x)} \quad \forall t > 0 \text{ and } x \in (0,\pi).$$

This can only be possible if both sides of the above equation are constant. This gives us the following ODEs for F and G:

$$\dot{G} + (5-k)G \;=\; 0, \tag{3}$$
$$F'' - kF \;=\; 0, \tag{4}$$

where k is a yet-undetermined constant.

As $u \not\equiv 0$, the boundary conditions imply that

$$F'(0) \;=\; 0 \;=\; F'(\pi).$$

The equation (4) has three different types of solutions depending on the sign of k

Case 1. *When $k > 0$.*

In this situation, $F(x) = C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x}$. If F has to satisfy the boundary conditions at $x = 0, \pi$, then a routine argument shows that $C_1 = C_2 = 0$. Thus, the case $k > 0$ does not yield any non-trivial solutions.

Case 2. *When $k = 0$.*

In this situation, $F(x) = C_1 x + C_2$, whence $F'(x) = C_1$. The boundary conditions imply that $C_1 = 0$, *but the constant C_2 can be non-zero.* Thus, we now need to consider the equation that determines G. As $k = 0$, (3) implies that $G(t) = Ae^{-5t}$, where A is some constant. Thus

$$u_0(x,t) \;=\; A_0 e^{-5t} \tag{5}$$

is a solution of the PDE $(*)$ of the product form satisfying the boundary conditions, where A_0 is an undetermined constant.

Case 3. *When* $k < 0$.

In this situation, it is notationally simpler to write $k = -p^2$. Then, $F(x) = C_1 \cos(px) + C_2 \sin(px)$. The boundary conditions give us the equations

$$pC_2 = 0, \quad -pC_1 \sin(p\pi) + pC_2 \cos(p\pi) = 0.$$

In the present case, $p \neq 0$. Hence, $C_2 = 0$, and we are faced with the condition $\sin(p\pi) = 0$. This implies

$$p\pi = n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$$

As $p \neq 0$, and as $\cos(-px) = \cos(px)$, it suffices to only consider $p = 1, 2, 3, \ldots$ Corresponding to each of these values of p, $k = -n^2$, and solving (3) yields $G(t) = Ae^{-(n^2+5)t}$. Thus, corresponding to each n, we have the solution

$$u_n(x,t) = A_n \cos(nx) e^{-(5+n^2)t}, \quad n = 1, 2, 3, \ldots, \tag{6}$$

where A_n is an undetermined constant.

All possible solutions of $(*)$ of the product form that satisfy the given boundary conditions are given by (5) and (6).

b) For the given problem, we can use superposition to look for a solution of the form

$$u(x,t) = A_0 e^{-5t} + \sum_{n=1}^{\infty} A_n \cos(nx) e^{-(5+n^2)t}.$$

Imposing the initial condition gives us

$$A_0 + \sum_{n=1}^{\infty} A_n \cos(nx) = \cos^2\left(\frac{x}{2}\right) - 2\cos(5x). \tag{7}$$

We use a half-range expansion, with period 2π, for the function on the right-hand side, and the above equation suggests a Fourier cosine series. But, rather than setting up the integrals to determine the Fourier coefficients, we recall the trigonometric identity:

$$\cos^2\left(\frac{x}{2}\right) = \frac{1+\cos(x)}{2}.$$

By (7) and orthogonality, $A_n = 0$ for all n except $n = 0, 1$ and 5. In the latter case: $A_0 = 1/2$, $A_1 = 1/2$ and $A_5 = -2$. Hence

$$u(x,t) = \frac{e^{-5t}}{2} + \frac{\cos(x)e^{-6t}}{2} - 2\cos(5x)e^{-30t}$$

is the desired solution.

Problem 5 (Problem 4 for Math 4M)

a) We use the following approximation of the Laplace operator:

$$\Delta u(x,y) = \frac{1}{h^2}\left(u(x+h,y)+u(x-h,y)+u(x,y+h)+u(x,y-h)-4u(x,y)\right).$$

Combining this with the central difference approximation of u_x given in the problem, we obtain the following difference scheme for the given PDE:

$$\begin{aligned} 1 = {} & \frac{1}{h^2}\left(u(x+h,y)+u(x-h,y)+u(x,y+h)+u(x,y-h)-4u(x,y)\right) \\ & + \frac{1}{2h}\left(u(x+h,y)-u(x-h,y)\right). \end{aligned}$$

(With the notation $u_{i,j} \approx u(i,j)$ — i.e. the approximation of the solution to the Dirichlet problem at the grid-points — the above can be expressed as:

$$u_{i,j} = \frac{3}{8}u_{i+1,j} + \frac{1}{8}u_{i-1,j} + \frac{1}{4}u_{i,j+1} + \frac{1}{4}u_{i,j-1} - \frac{1}{4}.\ )$$

b) The unknown quantities are then $u_{1,1}, u_{2,1}, u_{2,1}$ and $u_{2,2}$ while the other $u_{i,j}$s are given by the prescribed boundary values. One iteration with the Gauss-Seidel method, with the starting values equal to 1, yields:

$$u_{1,1} = \frac{3}{8}u_{2,1} + \frac{1}{8}u_{0,1} + \frac{1}{4}u_{1,2} + \frac{1}{4}u_{1,0} - \frac{1}{4} = \frac{3}{8} + \frac{1}{8} = \frac{1}{2},$$

$$u_{2,1} = \frac{3}{8}u_{3,1} + \frac{1}{8}u_{1,1} + \frac{1}{4}u_{2,2} + \frac{1}{4}u_{2,0} - \frac{1}{4} = 2\times\frac{3}{8} + \frac{1}{8}\times\frac{1}{2} + \frac{1}{4} + 0 - \frac{1}{4} = \frac{13}{16},$$

$$u_{1,2} = \frac{3}{8}u_{2,2} + \frac{1}{8}u_{0,2} + \frac{1}{4}u_{1,3} + \frac{1}{4}u_{1,1} - \frac{1}{4} = \frac{3}{8} + \frac{1}{8} + 0 + \frac{1}{4}\times\frac{1}{2} - \frac{1}{4} = \frac{3}{8},$$

and

$$u_{2,2} = \frac{3}{8}u_{3,2} + \frac{1}{8}u_{1,2} + \frac{1}{4}u_{2,3} + \frac{1}{4}u_{2,1} - \frac{1}{4} = 2\times\frac{3}{8} + \frac{1}{8}\times\frac{3}{8} + 0 + \frac{1}{4}\times\frac{13}{16} - \frac{1}{4} = \frac{3}{4}.$$

These are the approximations of $u(1,1), u(2,1), u(1,2)$ and $u(2,2)$ sought for.

Problem 6

a) We need to calculate Lagrange's interpolation polynomial. With the notation

$$\begin{aligned} P_1 &= (x-\tfrac{1}{2})(x-1)(x-\tfrac{3}{2})(x-2), \\ P_2 &= x(x-1)(x-\tfrac{3}{2})(x-2), \\ P_3 &= x(x-\tfrac{1}{2})(x-\tfrac{3}{2})(x-2), \\ P_4 &= x(x-\tfrac{1}{2})(x-1)(x-2), \\ P_5 &= (x-\tfrac{1}{2})(x-1)(x-\tfrac{3}{2}), \end{aligned}$$

the interpolating polynomial is given by

$$-11P_1(x)/P_1(0) - 7P_2(x)/P_2(\frac{1}{2}) + 3P_3(x)/P_3(1) + 25P_4(x)/P_4(\frac{3}{2}) + 65P_5(x)/P_5(2).$$

This simplifies to

$$P(x) = 8x^3 + 6x - 11.$$

Alternatively, **we can use Newton's divided-difference formula** for Lagrange's interpolation polynomial.

b) Since Simpson's method is *exact* for polynomials of degree less than or equal to three, the error is zero.

Problem 7 (Only for Math 4M, Problem 5 for 4M)

a) Heun's method for solving an ODE. The output is an approximation of $y(b)$ with intial value $y(a) = ya$, where y is a solution of

$$y' = e^{-y} - 1$$

on (a, b). The output would be $Q \approx 0.5300$. Follows from $k_1 = e^{-1} - 1$, $k_2 = e^{-e^{-1}} - 1$ and $Q = (k_1 + k_2)/2$.

b) Since $h = 1/n$ and the method is of second order we have $y \approx y_n + C/n^2$. Hence we obtain $y - y_{n=2} \approx 1/3 \times (P - Q) = 1/3 \times (0.4976 - 0.5300) \approx -0.01$.