Noregs teknisk–naturvitskaplege universitet
Institutt for matematiske fag

Fagleg kontakt under eksamen:
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EKSAMEN i MATEMATIKK 4N (TMA4130)

Laurdag 1. desember 2012
Tid: 09:00 – 13:00 Sensur: 3. januar 2013

Tillatne hjelpemidlar (Kode C):
Tillatt enkel kalkulator
Rottmann: *Matematisk formelsamling*
Formelark (følgjer med eksamen)

Alle svar må grunngjevast og det skal gå klart fram korleis svara er oppnådde.

Oppgåve 1 I denne oppgåva er a ein positiv konstant.

a) Finn den inverse Laplacetransformen for funksjonen

$$\frac{e^{-as}}{(s+1)(s-1)}.$$

b) Bruk Laplacetransformasjon for å løyse initialverdiproblemet:

$$\begin{aligned} y'' - y &= 2\delta(t-1), \\ y(0) = 1, \; y'(0) &= 1. \end{aligned}$$

Oppgåve 2 La f vere ein 2π-periodisk funksjon slik at

$$f(x) = \begin{cases} 0, & \text{hvis } -\pi \le x < 0, \\ x, & \text{hvis } 0 \le x < \pi. \end{cases}$$

a) Skisser grafen til f på intervallet $[-3\pi, 3\pi)$ og finn Fourierrekkja til f.

b) Ved å bruke resultatet frå **(a)**, finn summen av rekkja

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots$$

Oppgåve 3 For $a > 0$, finn Fouriertransformen til funksjonen $f(x) = xe^{-a|x|}$ og bruk dette til å rekne ut integralet

$$\int_{-\infty}^{\infty} \frac{w \sin(w)}{(w^2+1)^2}\, dw.$$

Oppgåve 4

a) Finn alle ikkje-trivielle løysingar (det vil seie andre løysingar enn løysinga $u \equiv 0$) av den partielle differensiallikninga

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 5u, \quad 0 < t,\ 0 < x < \pi, \qquad (*)$$

på forma $u(x,t) = F(x)G(t)$ som tilfredsstiller randkrava

$$u_x(0,t) = 0, \quad u_x(\pi,t) = 0 \ \text{ for alle } t \geq 0. \qquad (**)$$

b) Ved å bruke resultata frå **(a)**, finn løysinga av differensiallikninga $(*)$ som tilfredsstiller randkravet $(**)$ og som også tilfredsstiller initialkravet

$$u(x,0) = \cos^2\left(\frac{x}{2}\right) - 2\cos(5x), \quad 0 \leq x \leq \pi.$$

Oppgåve 5 Vi ser på følgande Dirichlet-problem på kvadratet med hjørne $(0,0)$, $(3,0)$, $(0,3)$ og $(3,3)$:

$$\begin{aligned} u_{xx} + u_{yy} + u_x &= 1, && \text{for } 0 < x, y < 3, \\ u(0,y) = 1, \quad u(3,y) &= 2, && \text{for } 0 < y < 3, \\ u(x,0) = 0, \quad u(x,3) &= 0, && \text{for } 0 < x < 3. \end{aligned}$$

a) La grid-punkta vere $(x_i, y_j) = (ih, jh)$ med $h = 1$. Ved å bruke den vanlege tilnærminga av u_{xx} og u_{yy}, og ei tilnærming av u_x ved hjelp av sentraldifferansar, altså

$$u_x(x,y) \approx \frac{1}{2h}[u(x+h,y) - u(x-h,y)],$$

skriv ned den numeriske algoritmen (numerical scheme) for dette problemet.

b) Set opp systemet av lineære likningar du får i (a) på ei form som kan brukast til å utføre Gauss–Seidel-iterasjonar. Gjer éin iterasjon av Gauss–Seidel for å få ei tilnærming av $u(1,1)$, $u(2,1)$, $u(1,2)$ og $u(2,2)$. Bruk startverdi 1 i alle indre punktei (interior points).

Oppgåve 6

a) Finn polynomet $P(x)$ av lågast mulig grad som interpolerer:

x_n	0	0.5	1	1.5	2
$P(x_n)$	-11	-7	3	25	65

b) La ϵ_S være feilen i Simpsons metode brukt på integralet

$$\int_0^2 P(x)\,dx$$

med verdiane gitt i tabellen over. Finn feilen ϵ_S *utan å bruke Simpsons integrasjonsformel eller å rekne ut integralet.* Grunngi svaret.

Formlar i numerikk

- La $p(x)$ vere eit polynom av grad $\leq n$ som interpolerer $f(x)$ i punkta $x_i, i = 0, 1, \ldots, n$. Dersom x og alle nodane ligg i intervallet $[a, b]$, så gjelder

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (x - x_i), \quad \xi \in (a, b).$$

- Newtons dividerte differansers interpolasjonspolynom $p(x)$ av grad $\leq n$:

$$\begin{aligned} p(x) = {} & f[x_0] + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] \\ & + \cdots + (x - x_0)(x - x_1) \ldots (x - x_{n-1}) f[x_0, \ldots, x_n] \end{aligned}$$

- Numerisk derivasjon:

$$\begin{aligned} f'(x) &= \frac{1}{h}[f(x+h) - f(x)] + \frac{1}{2} h f''(\xi) \\ f'(x) &= \frac{1}{2h}[f(x+h) - f(x-h)] - \frac{1}{6} h^2 f'''(\xi) \\ f''(x) &= \frac{1}{h^2}[f(x+h) - 2f(x) + f(x-h)] - \frac{1}{12} h^2 f^{(4)}(\xi) \end{aligned}$$

- Simpsons integrasjonsformel:

$$\int_{x_0}^{x_2} f(x)\, \mathrm{d}x \approx \frac{h}{3}(f_0 + 4f_1 + f_2)$$

- Newtons metode for likninga $f(x) = 0$ er gitt ved

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

- Iterative teknikkar for løysing av eit lineært likningssystem

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \qquad i = 1, 2, \ldots, n$$

$$\text{Jacobi:} \quad x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right)$$

$$\text{Gauss–Seidel:} \quad x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right)$$

- Heuns metode for løysing av $\mathbf{y}' = \mathbf{f}(x, \mathbf{y})$:

$$\begin{aligned} \mathbf{k}_1 &= h\mathbf{f}(x_n, \mathbf{y}_n) \\ \mathbf{k}_2 &= h\mathbf{f}(x_n + h, \mathbf{y}_n + \mathbf{k}_1) \\ \mathbf{y}_{n+1} &= \mathbf{y}_n + \frac{1}{2}(\mathbf{k}_1 + \mathbf{k}_2) \end{aligned}$$

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EXAM IN MATHEMATICS 4N (TMA4130)(TMA4135)

English
August 12, 2013
Time: 09.00-13.00

Permitted aids (Code C):
Approved calculator
Rottmann: *Matematisk formelsamling*
Formula sheet (handed out with the exam)

All answers must be justified and it should be clear how your assertions are obtained.

Problem 1 In this problem a is a positive constant.

a) Find the inverse Laplace transform of the function

$$\frac{e^{-as}}{s^2 - s}.$$

b) Use Laplace transforms to solve the initial-value problem:

$$\begin{aligned} y'' - y' &= 5\delta(t-2), \\ y(0) &= 0, \ \ y'(0) = -1. \end{aligned}$$

Problem 2 Let f be the function of period 2 such that $f(x) = 1 - x$ for $0 \le x < 2$. Find the Fourier series of f and use it to compute the sum of the series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Problem 3

a) Find all non-trivial solutions (i.e. solutions *other than* the solution $u \equiv 0$) of the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u, \quad 0 < t, \ 0 < x < 1, \qquad (*)$$

of the form $u(x,t) = F(x)G(t)$ that satisfy the boundary conditions

$$u_x(0,t) = 0, \quad u_x(1,t) = 0 \text{ for all } t \geq 0. \qquad (**)$$

b) Using your results from **(a)**, find the solution of the differential equation $(*)$ that satisfies the boundary condition $(**)$ and also satisfies the initial condition

$$u(x,0) = 2 + \cos^2\left(\frac{3\pi x}{2}\right) - 2\cos(3\pi x), \quad 0 \leq x \leq 1.$$

Problem 4 Consider the parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x, \quad \text{for } x \in (0,2), \ t > 0,$$

together with the initial condition

$$u(x,0) = x(3-x), \quad \text{for } 0 \leq x \leq 2,$$

and the boundary conditions

$$u(0,t) = 0, \quad u(2,t) = 2, \quad \text{for } t \geq 0.$$

Using a grid consisting of the points $t_j = jk$ and $x_i = ih$, write down an **explicit difference scheme** for the above problem. In other words, use a central difference approximation for $\frac{\partial^2 u}{\partial x^2}$ and a forward difference approximation for $\frac{\partial u}{\partial t}$.

Use this scheme with the step sizes $h = 0.5$ and $k = 0.1$ to obtain approximations for $u(0.5, 0.1)$, $u(1, 0.1)$ and $u(1.5, 0.1)$, where u is the solution of the above equation satisfying the given initial condition and boundary conditions.

Problem 5

a) For $a > 0$, find the Fourier transform of the function $g(x) = xe^{-ax}u(x)$, where $u(x)$ is the Heaviside function, i.e., $g(x)$ is given by

$$g(x) = \begin{cases} xe^{-ax}, & \text{for } x > 0, \\ 0, & \text{for } x \leq 0. \end{cases}$$

b) Using your result from part **(a)**, find the value of the integral

$$\int_{-\infty}^{\infty} \frac{1-w^2}{(1+w^2)^2} \cos(\pi w)\, dw.$$

Problem 6 You are given that the equation

$$x^4 - 4x + 1 = 0$$

has one solution s in the interval $(1/4, 1)$.

a) The above equation can be transformed into the equation $g(x) = x$ on the interval $(1/4, 1)$ for the following three choices of $g(x)$:

1) $g(x) = x^4 - 3x + 1$,
2) $g(x) = -1/(x^3 - 4)$,
3) $g(x) = \sqrt[4]{4x - 1}$.

Which of these will you choose to approximate s using the fixed-point iteration $x_{n+1} = g(x_n)$ and a starting value taken from the interval $(1/4, 1)$? Give reasons for your answer.

b) Use the g that you chose in **(a)**. With the starting value $x_0 = 0.5$, how many iterations are needed in order to calculate s correctly to 4 places of decimal. Perform the iterations.
Hint: Use

$$|x_{n+1} - s| \le \max_{x \in [1/4, 1]} |g'(x)||x_n - s|.$$

Numerical formulas

- Let $p(x)$ be the polynomial of degree $\leq n$ which coincides with $f(x)$ at points x_i, $i = 0, 1, \ldots, n$. Under the assumption that x and all the nodes x_j lie in the interval $[a, b]$, we have

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (x - x_i).$$

- Newton's divided difference interpolation formula $p(x)$ of degree $\leq n$:

$$\begin{aligned} p(x) = f[x_0] &+ (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] \\ &+ \cdots + (x - x_0)(x - x_1) \ldots (x - x_{n-1}) f[x_0, \ldots, x_n] \end{aligned}$$

- Numerical differentiation:

$$\begin{aligned} f'(x) &= \frac{1}{h}[f(x+h) - f(x)] + \frac{1}{2} h f''(\xi) \\ f'(x) &= \frac{1}{2h}[f(x+h) - f(x-h)] - \frac{1}{6} h^2 f'''(\xi) \\ f''(x) &= \frac{1}{h^2}[f(x+h) - 2f(x) + f(x-h)] - \frac{1}{12} h^2 f^{(4)}(\xi) \end{aligned}$$

- Simpson's rule of integration:

$$\int_{x_0}^{x_2} f(x)\, \mathrm{d}x \approx \frac{h}{3}(f_0 + 4f_1 + f_2)$$

- Newton's method for solving system of nonlinear equations $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ is given by the scheme

$$\begin{aligned} J^{(k)} \cdot \Delta \mathbf{x}^{(k)} &= -\mathbf{f}\left(\mathbf{x}^{(k)}\right) \\ \mathbf{x}^{(k+1)} &= \mathbf{x}^{(k)} + \Delta \mathbf{x}^{(k)}. \end{aligned}$$

- Iteration methods for solving systems of linear equations

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \qquad i = 1, 2, \ldots, n$$

$$\text{Jacobi:} \quad x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right)$$

$$\text{Gauss–Seidel:} \quad x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right)$$

- Heun's method for solving $\mathbf{y}' = \mathbf{f}(x, \mathbf{y})$:

$$\begin{aligned} \mathbf{k}_1 &= h\mathbf{f}(x_n, \mathbf{y}_n) \\ \mathbf{k}_2 &= h\mathbf{f}(x_n + h, \mathbf{y}_n + \mathbf{k}_1) \\ \mathbf{y}_{n+1} &= \mathbf{y}_n + \frac{1}{2}(\mathbf{k}_1 + \mathbf{k}_2) \end{aligned}$$

Table of some Laplace transforms

$f(t)$	$F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$t^n \ (n = 0, 1, 2, \ldots)$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$e^{at} \cos \omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
$e^{at} \sin \omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$

Table of some Fourier transforms

$f(x)$	$\hat{f}(w) = \mathcal{F}(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx$
$g(x) = f(ax)$	$\hat{g}(w) = \frac{1}{a} \hat{f}\left(\frac{w}{a}\right)$
$u(x) - u(x-a)$	$\frac{1}{\sqrt{2\pi}} \left(\frac{\sin aw}{w} - i \frac{1 - \cos aw}{w} \right)$
$u(x) e^{-x}$	$\frac{1}{\sqrt{2\pi}} \left(\frac{1}{1+w^2} - i \frac{w}{1+w^2} \right)$
e^{-ax^2}	$\frac{1}{\sqrt{2a}} e^{-\frac{w^2}{4a}}$
$e^{-a\|x\|}$	$\sqrt{\frac{2}{\pi}} \frac{a}{w^2+a^2}$