Norwegian University of Science and Technology Departement of Mathematical Sciences	TMA4130 Calculus 4N Fall 2013 **Exercise set 12**

1 Let $y(x)$ be a function that satisfies the ordinary differential equation

$$y'' = \cos(y) \tag{1}$$

with initial conditions $y(0) = 0$, $y'(0) = 1$.

a) Introduce suitable variables and write the equation (1) as a system of first order differential equations.

b) Use step size $h = 0.1$ and find an approximation to $y(0.2)$ by performing 2 steps with Heun's method.

2 Let $y = y(x)$ be the function satisfying the second order differential equation

$$y'' - (1 - y^2)y' + y = 0 \tag{2}$$

with initial conditions $y(0) = 2$, $y'(0) = 0$.

a) Rewrite equation (2) to a system of two first order ordinary differential equations. What is the initial condition for this system?

b) We want to solve the intial value problem

$$Y' = F(x, Y), \quad Y(x_0) = Y_0$$

numerically. Here Y is a column vector with the solutions as components. We have, among others, studied Euler's method for this problem in our lectures. Another method, known as "Backward Euler", is defined by

$$Y_{n+1} = Y_n + hF(x_{n+1}, Y_{n+1}), \tag{3}$$

where h denotes the step size, and we have used the notation $x_{n+1} = x_n + h$. We assume that we know the numerical solution Y_{n+1} in the $n + 1'$th step, and use the method (3) is used to define the numerical solution Y_{n+1} at step $n+1$. The method (3) is *implicit* since it evaluates the right hand side F in the *unknown* point Y_{n+1}. Because of this we must, in general, solve a non linear system of equations for every step to find the approximations Y_n for $n > 0$.

Let $y_{i,n}$ for $i = 1, 2$ denote component i of the numerical solution at step n. Use $h = 0.1$ and write down the non-linear system of equations for Y_1 based on "Backward Euler" for the system you deduced in **a)**. Use the known intial conditions Y_0 whenever possible.

c) Perform one iteration with Newton's method for the non-linear system of equations

$$10y_1 - y_2 - 20 = 0 \tag{4}$$
$$y_1 + (9 + y_1^2)y_2 = 0. \tag{5}$$

Use the starting values $y_1 = 2$ and $y_2 = 0$.

$\boxed{3}$ We consider the Poisson equation

$$u_{xx} + u_{yy} = -1, \qquad 0 \le x, y \le 1,$$

with homogeneous boundary conditions

$$u(x, 0) = 0, \quad u(x, 1) = 0, \quad u(0, y) = 0, \quad u(1, y) = 0.$$

We discretize the computational domain (the unit square) using a finite difference grid with grid spacing $h = 1/n$. The grid points are then given by $(x_i, y_j) = (ih, jh)$ for $0 \le i, j \le n$. The numerical approximation of $u(x_i, y_j)$ is denoted by $u_{i,j}$.

Use the five-point formula to construct the approximation $u_{i,j}$. Use $h = 1/3$ and derive the resulting system of algebraic equations (a 4×4 system) for the unknown vector $[u_{1,1}, u_{2,1}, u_{1,2}, u_{2,2}]^T$.

$\boxed{4}$ **a)** Vi vil løse ligningen

$$u_{xx} + u_{yy} = 27(x + y),$$

på enhetskvadratet $[0, 1] \times [0, 1]$ med randbetingelser

$$u(0, y) = u(x, 0) = 0, \qquad u(x, 1) = 3x, \qquad u(1, y) = 3y.$$

Finn en tilnærming til løsningen $u(x, y)$ ved å bruke sentraldifferanser for å approksimere u_{xx} og u_{yy}. La $h = 1/3$ være skrittlengden, og la gitteret være gitt av punktene $(x_i, y_j) = (ih, jh)$ for $i, j = 0, ..., 3$. Sett opp et system av ligninger for $U_{1,1}, U_{2,1}, U_{1,2}$ og $U_{2,1}$, der $U_{i,j} \approx u(x_i, y_j)$.

b) Utfør én Gauss-Seidel-iterasjon på systemet du fikk i oppgave a). Bruk som startvektor $\mathbf{x_0} = -(1, 1, 1, 1)$.

$\boxed{5}$ La problemet

$$\begin{array}{rll} (i) & u_t = u_{xx} & t > 0, 0 < x < 1 \\ (ii) & u(0, t) = u(1, t) = 0 & t \ge 0 \\ (iii) & u(x, 0) = 8x(1 - x) & 0 \le x \le 1 \end{array}$$

være gitt. Crank-Nicolsons skjema kan generelt gis ved

$$U_i^{n+1} - U_i^n = \frac{k}{2h^2}(U_{i-1}^n - 2U_i^n + U_{i+1}^n + U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}) \tag{6}$$

La $\Delta x = h = \frac{1}{4}, \Delta t = k = \frac{1}{16}$, og utled ligningsystemet for $U_i^1 = U(ih, k), i = 1, 2, 3,$ ved å bruke Crank-Nicolson som numerisk skjema.