Norwegian University of Science and Technology
Department of Mathematical Sciences

Exercise set 4

1 Use D'Alembert's method to find the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

for $t > 0$ and all $x \in \mathbb{R}$ with initial conditions

$$u(x,0) = \cos x \qquad \text{and} \qquad \frac{\partial u(x,0)}{\partial t} = xe^{-x^2}.$$

2 The PDE

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - \mu \frac{\partial u}{\partial t}$$

with constants $c > 0$ and $\mu > 0$ can be used for modeling the oscillations of a string with damping. We assume in the following that $c = 1$ and $\mu = 2$, and that the string has a length of π and satisfies the boundary conditions

$$u(0,t) = 0 \qquad \text{and} \qquad u(\pi,t) = 0.$$

a) Find all the solutions u of this equation with the given boundary conditions that are of the form $u(x,t) = F(x)G(t)$.

b) Find the solution of this equation with initial conditions

$$u(x,0) = x(\pi - x) \qquad \text{and} \qquad \frac{\partial u(x,0)}{\partial t} = 0.$$

You may use the fact that the function $f(x) = x(\pi - x)$ for $0 < x < \pi$ has the Fourier sine expansion

$$f(x) = \sum_{n=1}^{\infty} A_n \sin nx \qquad \text{with } A_n = \begin{cases} \dfrac{8}{\pi n^3} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

c) Verify that the function

$$u(x,t) = te^{-t}\sin(x)$$

solves the PDE with the given boundary conditions and the initial conditions

$$u(x,0) = 0 \qquad \text{and} \qquad \frac{\partial u(x,0)}{\partial t} = \sin(x).$$

Check whether you have also found this solution in part **a)** of this exercise.[1]

[1] If you have not, check possible exceptional cases in your solutions or/and revise the chapter on ordinary differential equations from your Mathematics 3 course.

3 We consider a long, thin gold bar of length $L = 1\text{m}$, one end of which is held at a constant temperature of $0°\text{C}$ whereas the other end is insulated. This can be modeled by the heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

with boundary conditions

$$u(0,t) = 0 \qquad \text{and} \qquad \frac{\partial u(1,t)}{\partial x} = 0.$$

Here $c^2 \approx 1.27 \times 10^{-4}\text{m}^2/\text{s}$ is the thermal diffusivity of gold.

a) Compute all solutions of this equation with the given boundary conditions that are of the form $u(x,t) = F(x)G(t)$.

b) Assume that the initial temperature distribution in the gold bar is given by

$$u(x,0) = 100\sin(\pi x/2).$$

(u is given in degree Celsius). How long does it take, until the insulated end of the gold bar has cooled down to $20°\text{C}$?