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Løsningsforslag — Øving 5

1 Siden funksjonen er jamn og med periode π (merk at $L = \frac{\pi}{2}$) er Fourierrekka ei cosinusrekke med koeffisienter $a_0 = \frac{\pi}{4}$, gjennomsnittsverdien, og

$$\begin{aligned}
a_n &= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} f(x)\cos 2nx dx \\
&= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} x\cos 2nx dx \\
&= \frac{4}{\pi}\left[\frac{1}{2n}x\sin 2nx + \frac{1}{(2n)^2}\cos 2nx\right]_0^{\frac{\pi}{2}} \\
&= \frac{4}{\pi}\left(\frac{1}{(2n)^2}\cos n\pi - 1\right) = \frac{4}{\pi}\frac{1}{n^2}\left((-1)^n - 1\right).
\end{aligned}$$

$$a_n = \begin{cases} 0 & \text{for} \quad n = 2,4,6... \\ \frac{-2}{\pi n^2} & \text{for} \quad n = 1,3,5... \end{cases}$$

De trigonometriske polynomene som gir minst kvadratfeil er

$$F_N(x) = a_0 + \sum_{n=1}^{N} a_n \cos 2nx,$$

og feilen

$$E_N = \int_{-L}^{L} f^2 - 2La_0^2 - L\sum_{n=1}^{N} a_n^2$$

$$E_N = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^2)\,dx - \frac{\pi}{2}\left(2\frac{\pi^2}{4^2} + \sum_{n=1}^{N} a_n^2\right) = \frac{\pi^3}{48} - \frac{\pi}{2}\sum_{n=1}^{N} a_n^2.$$

Ved hjelp av lommeregner får vi

$$\begin{aligned}
E_0 &= \frac{\pi^3}{48} = 0.645964098\ldots \\
E_1 = E_2 &= \frac{\pi^3}{48} - \frac{2}{\pi}\frac{1}{1^4} = 0.009344325\ldots \\
E_3 = E_4 &= \frac{\pi^3}{48} - \frac{2}{\pi}\left(\frac{1}{1^4} + \frac{1}{3^4}\right) = 0.001484822\ldots \\
E_5 = E_6 &= \frac{\pi^3}{48} - \frac{2}{\pi}\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4}\right) = 0.000466230\ldots \\
E_7 = E_8 &= \frac{\pi^3}{48} - \frac{2}{\pi}\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4}\right) = 0.000201083\ldots
\end{aligned}$$

2 Siden funksjonen er odde og med periode 2π er Fourierrekka ei sinusrekke med koeffisienter

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx \\ &= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\sin nx dx \\ &= \frac{2}{\pi}\left[\frac{1}{n^2}\sin nx - \frac{1}{n}x\cos nx\right]_0^{\frac{\pi}{2}} \\ &= \frac{2}{\pi}\left(\frac{1}{n^2}\sin n\frac{\pi}{2} - \frac{1}{n}\frac{\pi}{2}\cos n\frac{\pi}{2}\right) \\ &= \frac{2}{\pi}\left(\frac{1}{n^2}\sin n\frac{\pi}{2}\right) - \frac{1}{n}\cos n\frac{\pi}{2}. \end{aligned}$$

Vi har

n	1	2	3	4	5	6	7	8
$\sin n\frac{\pi}{2}$	1	0	-1	0	1	0	-1	0
$\cos n\frac{\pi}{2}$	0	-1	0	1	0	-1	0	1

Dermed blir

$$b_n = \begin{cases} \frac{2}{\pi}\frac{1}{n^2} & \text{når} \quad n = 4k+1, \\ \frac{1}{n} & \text{når} \quad n = 4k+2, \\ -\frac{2}{\pi}\frac{1}{n^2} & \text{når} \quad n = 4k+3, \\ -\frac{1}{n} & \text{når} \quad n = 4k. \end{cases}$$

De trigonometriske polynomene som gir minst kvadratfeil er

$$F_N(x) = \sum_{n=1}^{N} b_n \sin nx,$$

og feilen er

$$\begin{aligned} E_N &= \int_{-\pi}^{\pi} (f(x))^2\, dx - \pi\sum_{n=1}^{N} b_n^2 \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 dx - \pi\sum_{n=1}^{N} b_n^2 \\ &= \frac{\pi^3}{12} - \pi\sum_{n=1}^{N} b_n^2. \end{aligned}$$

Ved hjelp av lommeregner får vi

$$E_1 = \frac{\pi^3}{12} - \pi\left(\left(\frac{2}{\pi}\right)^2 \frac{1}{1^4}\right) = 1,310616846\ldots$$
$$E_2 = \frac{\pi^3}{12} - \pi\left(\left(\frac{2}{\pi}\right)^2 \frac{1}{1^4} + \frac{1}{2^2}\right) = 0,525218682\ldots$$
$$E_3 = \frac{\pi^3}{12} - \pi\left(\left(\frac{2}{\pi}\right)^2 \frac{1}{1^4} + \frac{1}{2^2} + \left(\frac{2}{\pi}\right)^2 \frac{1}{3^4}\right) = 0,509499675\ldots$$
$$E_4 = \frac{\pi^3}{12} - \pi\left(\left(\frac{2}{\pi}\right)^2 \frac{1}{1^4} + \frac{1}{2^2} + \left(\frac{2}{\pi}\right)^2 \frac{1}{3^4} + \frac{1}{4^2}\right) = 0,313150134\ldots.$$

3 Vi har

$$\begin{aligned} x^2 &= \frac{\pi^2}{3} - 4\left(\cos x - \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x - \frac{1}{16}\cos 4x + - \cdots\right) \\ &= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx, \qquad -\pi \le x \le \pi. \end{aligned}$$

Parsevals identitet gir da

$$2\left(\frac{\pi^2}{3}\right)^2 + \sum_{n=1}^{\infty}\left(4\frac{(-1)^n}{n^2}\right)^2 = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x^2\right)^2\,dx = \frac{2\pi^4}{5},$$
$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{1}{16}\left(2\cdot\frac{\pi^4}{5} - 2\cdot\frac{\pi^4}{9}\right) = \frac{\pi^4}{90} \qquad (\approx 1.082).$$

De fire første partialsummene er

$S_1 = 1, \quad S_2 = S_1 + 1/2^4 \approx 1.063, \quad S_3 = S_2 + 1/3^4 \approx 1.075, \quad S_4 = S_3 + 1/4^4 \approx 1.079.$

4 **a)** Fourierrekka er $\sum_{-\infty}^{\infty} c_n e^{inx}$, der $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-2x}e^{-inx}dx$.
Litt regning gir at $c_n = \frac{\sinh 2\pi}{\pi}(-1)^n\frac{1}{2+in}$. (Merk at $e^{in\pi} = e^{-in\pi} = (-1)^n$.)

b) Dersom vi splitter funksjonen $g(x)$ i en odde funksjon og en jamn funksjon så er $g(x) = e^x = \cosh x + \sinh x$ for $-\pi < x < \pi$. I tillegg er den jamne delen av den periodiske utvidelsen kontinuerlig for $x = \pi$. Vi ser på den jamne delen. Vi har

$$a_0 = c_0 = \frac{\sinh\pi}{\pi},$$
$$a_n = c_n + c_{-n} = (-1)^n\frac{\sinh\pi}{\pi}\frac{2}{1+n^2}.$$

Altså er

$$\cosh x = \frac{\sinh\pi}{\pi} + 2\frac{\sinh\pi}{\pi}\sum_{1}^{\infty}(-1)^n\frac{1}{1+n^2}\cos nx \quad \text{for} \quad x\in[-\pi,\pi].$$

Ved å sette in $x = 0$ får vi

$$1 = \frac{\sinh \pi}{\pi} + 2\frac{\sinh \pi}{\pi}\sum_{1}^{\infty}(-1)^n \frac{1}{1+n^2}.$$

Litt algebra viser at

$$\sum_{1}^{\infty}(-1)^n \frac{1}{1+n^2} = \frac{\pi - \sinh \pi}{2\sinh \pi} \quad \text{og} \quad \sum_{-\infty}^{\infty}(-1)^n \frac{1}{1+n^2} = \frac{\pi}{\sinh \pi}.$$

Ved å sette in $x = \pi$ får vi

$$\cosh \pi = \frac{\sinh \pi}{\pi} + 2\frac{\sinh \pi}{\pi}\sum_{1}^{\infty}\frac{1}{1+n^2}.$$

Litt algebra viser da at

$$\sum_{1}^{\infty}\frac{1}{1+n^2} = \frac{\pi \cosh \pi - \sinh \pi}{2\sinh \pi} \quad \text{og} \quad \sum_{-\infty}^{\infty}\frac{1}{1+n^2} = \frac{\pi \cosh \pi}{\sinh \pi}.$$