**THEOREM 3**

**Existence Theorem for Laplace Transforms**

*If $f(t)$ is defined and piecewise continuous on every finite interval on the semi-axis $t \geqq 0$ and satisfies (2) for all $t \geqq 0$ and some constants $M$ and $k$, then the Laplace transform $\mathcal{L}(f)$ exists for all $s > k$.*

**PROOF** Since $f(t)$ is piecewise continuous, $e^{-st}f(t)$ is integrable over any finite interval on the $t$-axis. From (2), assuming that $s > k$ (to be needed for the existence of the last of the following integrals), we obtain the proof of the existence of $\mathcal{L}(f)$ from

$$|\mathcal{L}(f)| = \left| \int_0^\infty e^{-st} f(t)\, dt \right| \leqq \int_0^\infty |f(t)| e^{-st}\, dt \leqq \int_0^\infty M e^{kt} e^{-st}\, dt = \frac{M}{s-k}. \quad \blacksquare$$

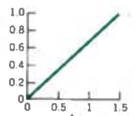
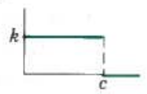
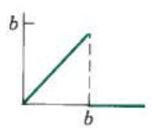
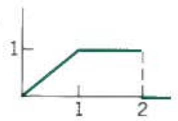
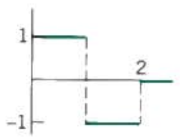
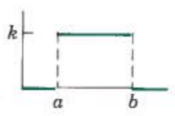
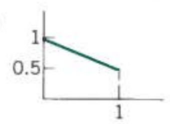
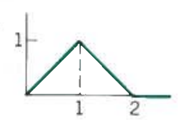
Note that (2) can be readily checked. For instance, $\cosh t < e^t$, $t^n < n! e^t$ (because $t^n/n!$ is a single term of the Maclaurin series), and so on. A function that does not satisfy (2) for any $M$ and $k$ is $e^{t^2}$ (take logarithms to see it). We mention that the conditions in Theorem 3 are sufficient rather than necessary (see Prob. 22).

**Uniqueness.** If the Laplace transform of a given function exists, it is uniquely determined. Conversely, it can be shown that if two functions (both defined on the positive real axis) have the same transform, these functions cannot differ over an interval of positive length, although they may differ at isolated points (see Ref. [A14] in App. 1). Hence we may say that the inverse of a given transform is essentially unique. In particular, if two *continuous* functions have the same transform, they are completely identical.

## PROBLEM SET 6.1

### 1–16 LAPLACE TRANSFORMS

Find the transform. Show the details of your work. Assume that $a, b, \omega, \theta$ are constants.

1. $2t + 8$
2. $(a - bt)^2$
3. $\cos 2\pi t$
4. $\cos^2 \omega t$
5. $e^{3t} \sinh t$
6. $e^{-t} \sinh 4t$
7. $\cos(\omega t + \theta)$
8. $1.5 \sin(3t - \pi/2)$
9. 
10. 
11. 
12. 
13. 
14. 
15. 
16. 

### 17–24 SOME THEORY

17. **Table 6.1.** Convert this table to a table for finding inverse transforms (with obvious changes, e.g., $\mathcal{L}^{-1}(1/s^n) = t^{n-1}/(n-1)$, etc).
18. Using $\mathcal{L}(f)$ in Prob. 10, find $\mathcal{L}(f_1)$, where $f_1(t) = 0$ if $t \leqq 2$ and $f_1(t) = 1$ if $t > 2$.
19. **Table 6.1.** Derive formula 6 from formulas 9 and 10.
20. **Nonexistence.** Show that $e^{t^2}$ does not satisfy a condition of the form (2).
21. **Nonexistence.** Give simple examples of functions (defined for all $t \geqq 0$) that have no Laplace transform.
22. **Existence.** Show that $\mathcal{L}(1/\sqrt{t}) = \sqrt{\pi/s}$. [Use (30) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ in App. 3.1.] Conclude from this that the conditions in Theorem 3 are sufficient but not necessary for the existence of a Laplace transform.



23. **Change of scale.** If $\mathcal{L}(f(t)) = F(s)$ and $c$ is any positive constant, show that $\mathcal{L}(f(ct)) = F(s/c)/c$ (*Hint:* Use (1).) Use this to obtain $\mathcal{L}(\cos \omega t)$ from $\mathcal{L}(\cos t)$.
24. **Inverse transform.** Prove that $\mathcal{L}^{-1}$ is linear. *Hint:* Use the fact that $\mathcal{L}$ is linear.

### 25–32 INVERSE LAPLACE TRANSFORMS

Given $F(s) = \mathcal{L}(f)$, find $f(t)$. $a, b, L, n$ are constants. Show the details of your work.

25. $\dfrac{0.2s + 1.4}{s^2 + 1.96}$
26. $\dfrac{5s + 1}{s^2 - 25}$
27. $\dfrac{s}{L^2 s^2 + 1/4\, \pi^2}$
28. $\dfrac{1}{(s + \sqrt{2})(s - \sqrt{3})}$
29. $\dfrac{2}{s^4} - \dfrac{48}{s^6}$
30. $\dfrac{4s + 32}{s^2 - 16}$
31. $\dfrac{-s + 11}{s^2 - 2s - 3}$
32. $\dfrac{1}{(s + a)(s + b)}$

### 33–45 APPLICATION OF s-SHIFTING

In Probs. 33–36 find the transform. In Probs. 37–45 find the inverse transform. Show the details of your work.

33. $t^3 e^{-2t}$
34. $ke^{-at} \cos \omega t$
35. $2e^{-1/2t} \sin 4\pi t$
36. $\sinh t \cos t$
37. $\dfrac{2\pi}{(s + \pi)^3}$
38. $\dfrac{6}{(s + 1)^3}$
39. $\dfrac{90}{(s + \sqrt{3})^6}$
40. $\dfrac{4}{s^2 - 2s - 3}$
41. $\dfrac{\pi}{s^2 + 4s\pi + 3\pi^2}$
42. $\dfrac{a_0}{s + 1} + \dfrac{a_1}{(s + 1)^2} + \dfrac{a_2}{(s + 1)^3}$
43. $\dfrac{6s + 7}{2s^2 + 4s + 10}$
44. $\dfrac{a(s + k) + b\pi}{(s + k)^2 + \pi^2}$
45. $\dfrac{k_0}{s} + \dfrac{k_1}{(s - a)^2}$

# 6.2 Transforms of Derivatives and Integrals. ODEs

The Laplace transform is a method of solving ODEs and initial value problems. The crucial idea is that ***operations of calculus on functions are replaced by operations of algebra on transforms***. Roughly, *differentiation* of $f(t)$ will correspond to *multiplication* of $\mathcal{L}(f)$ by $s$ (see Theorems 1 and 2) and *integration* of $f(t)$ to *division* of $\mathcal{L}(f)$ by $s$. To solve ODEs, we must first consider the Laplace transform of derivatives. You have encountered such an idea in your study of logarithms. Under the application of the natural logarithm, a product of numbers becomes a sum of their logarithms, a division of numbers becomes their difference of logarithms (see Appendix 3, formulas (2), (3)). To simplify calculations was one of the main reasons that logarithms were invented in pre-computer times.

**THEOREM 1**

**Laplace Transform of Derivatives**

*The transforms of the first and second derivatives of $f(t)$ satisfy*

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0) \quad (1)$$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0). \quad (2)$$

*Formula (1) holds if $f(t)$ is continuous for all $t \geqq 0$ and satisfies the growth restriction (2) in Sec. 6.1 and $f'(t)$ is piecewise continuous on every finite interval on the semi-axis $t \geqq 0$. Similarly, (2) holds if $f$ and $f'$ are continuous for all $t \geqq 0$ and satisfy the growth restriction and $f''$ is piecewise continuous on every finite interval on the semi-axis $t \geqq 0$.*