and using (1), we obtain

$$F'(s)\} = \mathcal{L}^{-1}\left\{\frac{2s}{s^2+\omega^2} - \frac{2}{s}\right\} = 2\cos\omega t - 2 = -tf(t).$$

) is $f(t) = 2(1 - \cos\omega t)/t$. This agrees with formula 42 in Sec. 6.9.

$$\frac{2s}{+\omega^2} - \frac{2}{s}, \qquad \text{then} \qquad g(t) = \mathcal{L}^{-1}(G) - 2(\cos\omega t - 1).$$

agreement with the answer just obtained,

$$\frac{s^2+\omega^2}{s^2}\Bigg\} = \mathcal{L}^{-1}\left\{\int_s^\infty G(s)\,ds\right\} = -\frac{g(t)}{t} = \frac{2}{t}(1 - \cos\omega t),$$

s the lower limit of integration.
formula 43 in Sec. 6.9,

$$\mathcal{L}^{-1}\left\{\ln\left(1 - \frac{a^2}{s^2}\right)\right\} = \frac{2}{t}(1 - \cosh at).$$ ■

ODEs with Variable Coefficients

d to solve certain ODEs with variable coefficients. The idea is this.
$y') = sY - y(0)$ (see Sec. 6.2). Hence by (1),

$$\mathcal{L}(ty') = -\frac{d}{ds}[sY - y(0)] = -Y - s\frac{dY}{ds}.$$

$' - sy(0) - y'(0)$ and by (1)

$$-\frac{d}{ds}[s^2Y - sy(0) - y'(0)] = -2sY - s^2\frac{dY}{ds} + y(0).$$

oefficients such as $at + b$, the subsidiary equation is a first-order
netimes simpler than the given second-order ODE. But if the latter
$bt + c$, then two applications of (1) would give a second-order
ows that the present method works well only for rather special
fficients. An important ODE for which the method is advantageous

guerre Polynomials

$$ty'' + (1 - t)y' + ny = 0.$$

) with $n = 0, 1, 2, \cdots$. From (7)–(9) we get the subsidiary equation

$$-s^2\frac{dY}{ds} + y(0)\Big] + sY - y(0) - \left(-Y - s\frac{dY}{ds}\right) + nY = 0.$$

Simplification gives

$$(s - s^2)\frac{dY}{ds} + (n + 1 - s)Y = 0.$$

Separating variables, using partial fractions, integrating (with the constant of integration taken to be zero), and taking exponentials, we get

$$\text{(10*)} \qquad \frac{dY}{Y} = -\frac{n+1-s}{s-s^2}\,ds = \left(\frac{n}{s-1} - \frac{n+1}{s}\right)ds \qquad \text{and} \qquad Y = \frac{(s-1)^n}{s^{n+1}}.$$

We write $l_n = \mathcal{L}^{-1}(Y)$ and prove **Rodrigues's formula**

$$\text{(10)} \qquad l_0 = 1, \qquad l_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t}), \qquad n = 1, 2, \cdots.$$

These are polynomials because the exponential terms cancel if we perform the indicated differentiations. They are called **Laguerre polynomials** and are usually denoted by L_n (see Problem Set 5.7, but we continue to reserve capital letters for transforms). We prove (10). By Table 6.1 and the first shifting theorem (*s*-shifting),

$$\mathcal{L}(t^n e^{-t}) = \frac{n!}{(s+1)^{n+1}}, \qquad \text{hence by (3) in Sec. 6.2} \qquad \mathcal{L}\left\{\frac{d^n}{dt^n}(t^n e^{-t})\right\} = \frac{n!s^n}{(s+1)^{n+1}}$$

because the derivatives up to the order $n - 1$ are zero at 0. Now make another shift and divide by $n!$ to get [see (10) and then (10*)]

$$\mathcal{L}(l_n) = \frac{(s-1)^n}{s^{n+1}} = Y.$$ ■

PROBLEM SET 6.6

1. REVIEW REPORT. Differentiation and Integration of Functions and Transforms. Make a draft of these four operations from memory. Then compare your draft with the text and write a 2- to 3-page report on these operations and their significance in applications.

2–11 TRANSFORMS BY DIFFERENTIATION

Showing the details of your work, find $\mathcal{L}(f)$ if $f(t)$ equals:

2. $3t \sinh 4t$

3. $\frac{1}{4}e^{-2t}$

4. $te^{-t}\cos t$

5. $t \sin \omega t$

6. $t^2 \sin 3t$

7. $t^2 \sinh 2t$

8. $te^{-kt} \sin t$

9. $\frac{1}{2}t^2 \cos \frac{\pi}{2}t$

10. $t^n e^{kt}$

11. $\frac{1}{2}\sin 2\pi t$

12. CAS PROJECT. Laguerre Polynomials. (a) Write a CAS program for finding $l_n(t)$ in explicit form from (10). Apply it to calculate $l_0, \cdots, l_{10}$. Verify that $l_0, \cdots, l_{10}$ satisfy Laguerre's differential equation (9).

(b) Show that

$$l_n(t) = \sum_{m=0}^{n} \frac{(-1)^m}{m!}\binom{n}{m}t^m$$

and calculate $l_0, \cdots, l_{10}$ from this formula.

(c) Calculate $l_0, \cdots, l_{10}$ recursively from $l_0 = 1$, $l_1 = 1 - t$ by

$$(n + 1)l_{n+1} = (2n + 1 - t)l_n - nl_{n-1}.$$

(d) A **generating function** (definition in Problem Set 5.2) for the Laguerre polynomials is

$$\sum_{n=0}^{\infty} l_n(t)x^n = (1 - x)^{-1}e^{tx/(x-1)}.$$

Obtain $l_0, \cdots, l_{10}$ from the corresponding partial sum of this power series in x and compare the l_n with those in (a), (b), or (c).

13. CAS EXPERIMENT. Laguerre Polynomials. Experiment with the graphs of $l_0, \cdots, l_{10}$, finding out empirically how the first maximum, first minimum, $\cdots$ is moving with respect to its location as a function of n. Write a short report on this.