PROBLEM SET 9.7

1–6 CALCULATION OF GRADIENTS

Find grad f. Graph some level curves f = const. Indicate ∇f by arrows at some points of these curves.

1. $f = (x - 1)(4y - 2)$
2. $f = 2x^2 + 5y^2$
3. $f = x/y$
4. $f = (x - 2)^2 + (2y + 4)^2$
5. $f = x^5 + y^5$
6. $f = (x^2 + y^2)/(x^2 - y^2)$

7–10 USEFUL FORMULAS FOR GRADIENT AND LAPLACIAN

Prove and illustrate by an example.

7. $\nabla(f^n) = nf^{n-1}\nabla f$
8. $\nabla(fg) = f\nabla g + g\nabla f$
9. $\nabla(f/g) = (1/g^2)(g\nabla f - f\nabla g)$
10. $\nabla^2(fg) = g\nabla^2 f + 2\nabla f \cdot \nabla g + f\nabla^2 g$

11–15 USE OF GRADIENTS. ELECTRIC FORCE

The force in an electrostatic field given by $f(x, y, z)$ has the direction of the gradient. Find ∇f and its value at P.

11. $f = xy$, P: (3, −4,)
12. $f = x/(x^2 + y^2)$, P: (1, 1)
13. $f = \ln(x^2 + y^2)$, P: (4, 2)
14. $f = (x^2 + y^2 + z^2)^{-1/2}$ P: (12, 0, 16)
15. $f = 2x^2 + 4y^2 + 9z^2$, P: (−1, 2, −4)
16. For what points P: (x, y, z) does ∇f with $f = 25x^2 + 9y^2 + 16z^2$ have the direction from P to the origin?
17. Same question as in Prob. 16 when $f = 4x^2 + 25y^2$.

18–23 VELOCITY FIELDS

Given the velocity potential f of a flow, find the velocity $\mathbf{v} = \nabla f$ of the field and its value $\mathbf{v}(P)$ at P. Sketch $\mathbf{v}(P)$ and the curve f = const passing through P.

18. $f = x^2 - 6x - y^2$, P: (−1, 5)
19. $f = \cos x \cosh y$, P: $(\frac{1}{2}\pi, \ln 2)$
20. $f = x(1 + (x^2 + y^2)^{-1})$, P: (1, 1)
21. $f = e^x \cos y$, P: $(1, \frac{1}{2}\pi)$
22. At what points is the flow in Prob. 21 directed vertically upward?
23. At what points is the flow in Prob. 21 horizontal?

24–27 HEAT FLOW

Experiments show that in a temperature field, heat flows in the direction of maximum decrease of temperature T. Find this direction in general and at the given point P. Sketch that direction at P as an arrow.

24. $T = 3x^2 - 2y^2$, P: (2.5, 1.8)
25. $T = z/(x^2 + y^2)$, P: (0, 1, 2)
26. $T = x^2 + y^2 + 4z^2$, P: (2, −1, 2)
27. **CAS PROJECT. Isotherms.** Graph some curves of constant temperature ("isotherms") and indicate directions of heat flow by arrows when the temperature equals (**a**) $x^3 - 3xy^2$, (**b**) $\sin x \sinh y$, and (**c**) $e^x \cos y$.
28. **Steepest ascent.** If $z(x, y) = 3000 - x^2 - 9y^2$ [meters] gives the elevation of a mountain at sea level, what is the direction of steepest ascent at P: (4, 1)?
29. **Gradient.** What does it mean if $|\nabla f(P)| > |\nabla f(Q)|$ at two points P and Q in a scalar field?

30–35 CURVE AND SURFACE NORMAL VECTORS

Find a normal vector of the curve or surface at the given point P. Make a sketch.

30. $4x^2 + 9y^2 = 72$, P: $(2, \sqrt{56}/3)$
31. $16x^2 - y^2 = 399$, P: (5, 1)
32. $ax + by + cz = d$, any P
33. $6x^2 + 2y^2 + z^2 = 225$, P: (5, 5, 5)
34. $x^4 + y^4 + z^4 = 273$, P: (2, 1, 4)
35. $z - x^2 - y^2 = -638$, P: (3, 4, 25)

36–42 DIRECTIONAL DERIVATIVE of f at P in the direction of $\mathbf{a}$. Find it and make a sketch.

36. $f = 2x^2 + 2y^2$, P: (3, 3), $\mathbf{a} = [-1, -3]$
37. $f = x - y$, P: (4, 5), $\mathbf{a} = [2, 1]$
38. $f = x^2 + y^2 + z^2$, P: (2, 2, −1), $\mathbf{a} = [1, 1, 3]$
39. $f = 1/\sqrt{x^2 + y^2 + z^2}$, P: (3, 0, 4), $\mathbf{a} = [1, 1, 1]$
40. $f = \ln(x^2 + y^2)$, P: (3, 0), $\mathbf{a} = [1, -1]$
41. $f = xyz$, P: (−1, 1, 3), $\mathbf{a} = [1, -1, 2]$
42. $f = 4x^2 + 25y^2 + 9z^2$, P: (5, 0, 0), $\mathbf{a} = [0, 1, 1]$

43–45 POTENTIALS of a given vector field—if they exist!—can be obtained by a method to be shown in Sec. 10.2. In simpler cases, use inspection. Find a potential f for given $\mathbf{v}$ = grad f.

43. $\mathbf{v} = [yz, xz, xy]$
44. $\mathbf{v} = [ye^x, e^x, z^2]$
45. $\mathbf{v} = [v_1(x), v_2(y), v_3(z)]$