EXAMPLE 8 **Secant Method**

Find the positive solution of $f(x) = x - 2\sin x = 0$ by the secant method, starting from $x_0 = 2, x_1 = 1.9$.

Solution. Here, (10) is

$$x_{n+1} = x_n - \frac{(x_n - 2\sin x_n)(x_n - x_{n-1})}{x_n - x_{n-1} + 2(\sin x_{n-1} - \sin x_n)} = x_n - \frac{N_n}{D_n}.$$

Numeric values are:

n	x_{n-1}	x_n	N_n	D_n	$x_{n+1} - x_n$
1	2.000000	1.900000	−0.000740	−0.174005	−0.004253
2	1.900000	1.895747	−0.000002	−0.006986	−0.000252
3	1.895747	1.895494	0		0

$x_3 = 1.895494$ is exact to 6D. See Example 4. ■

Summary of Methods. The methods for computing solutions s of $f(x) = 0$ with given continuous (or differentiable) $f(x)$ start with an initial approximation x_0 of s and generate a sequence $x_1, x_2, \cdots$ by **iteration**. **Fixed-point methods** solve $f(x) = 0$ written as $x = g(x)$, so that s is a *fixed point* of g, that is, $s = g(s)$. For $g(x) = x - f(x)/f'(x)$ this is **Newton's method**, which, for good x_0 and simple zeros, converges quadratically (and for multiple zeros linearly). From Newton's method the **secant method** follows by replacing $f'(x)$ by a difference quotient. The ***bisection method*** and the ***method of false position*** in Problem Set 19.2 always converge, but often slowly.

PROBLEM SET 19.2

1–11 FIXED-POINT ITERATION

Solve by fixed-point iteration and answer related questions where indicated. Show details.

1. **Monotone sequence.** Why is the sequence in Example 1 monotone? Why not in Example 2?
2. Do the iterations (b) in Example 2. Sketch a figure similar to Fig. 427. Explain what happens.
3. $f = x - 0.5\cos x = 0, \quad x_0 = 1$. Sketch a figure.
4. $f = x - \operatorname{cosec} x$ the zero near $x = 1$.
5. Sketch $f(x) = x^3 - 5.00x^2 + 1.01x + 1.88$, showing roots near ± 1 and 5. Write $x = g(x) = (5.00x^2 - 1.01x + 1.88)/x^2$. Find a root by starting from $x_0 = 5, 4, 1, -1$. Explain the (perhaps unexpected) results.
6. Find a form $x = g(x)$ of $f(x) = 0$ in Prob. 5 that yields convergence to the root near $x = 1$.
7. Find the smallest positive solution of $\sin x = e^{-x}$.
8. **Elasticity.** Solve $x \cosh x = 1$. (Similar equations appear in vibrations of beams; see Problem Set 12.3.)
9. **Drumhead. Bessel functions.** A partial sum of the Maclaurin series of $J_0(x)$ (Sec. 5.5) is $f(x) = 1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{1}{2304}x^6$. Conclude from a sketch that $f(x) = 0$ near $x = 2$. Write $f(x) = 0$ as $x = g(x)$ (by dividing $f(x)$ by $\frac{1}{4}x$ and taking the resulting x-term to the other side). Find the zero. (See Sec. 12.10 for the importance of these zeros.)
10. **CAS EXPERIMENT. Convergence.** Let $f(x) = x^3 + 2x^2 - 3x - 4 = 0$. Write this as $x = g(x)$, for g choosing (1) $(x^3 - f)^{1/3}$, (2) $(x^2 - \frac{1}{2}f)^{1/2}$, (3) $x + \frac{1}{3}f$, (4) $(x^3 - f)/x^2$, (5) $(2x^2 - f)/(2x)$, and (6) $x - f/f'$ and in each case $x_0 = 1.5$. Find out about convergence and divergence and the number of steps to reach 6S-values of a root.
11. **Existence of fixed point.** Prove that if g is continuous in a closed interval I and its range lies in I, then the equation $x = g(x)$ has at least one solution in I. Illustrate that it may have more than one solution in I.

12–18 NEWTON'S METHOD

Apply Newton's method (6S-accuracy). First sketch the function(s) to see what is going on.

12. **Cube root.** Design a Newton iteration. Compute $\sqrt[3]{7}, x_0 = 2$.
13. $f = 2x - \cos x, \quad x_0 = 1$. Compare with Prob. 3.