EXAMPLE 8 **Secant Method**

Find the positive solution of $f(x) = x - 2\sin x = 0$ by the secant method, starting from $x_0 = 2, x_1 = 1.9$.

Solution. Here, (10) is

$$x_{n+1} = x_n - \frac{(x_n - 2\sin x_n)(x_n - x_{n-1})}{x_n - x_{n-1} + 2(\sin x_{n-1} - \sin x_n)} = x_n - \frac{N_n}{D_n}.$$

Numeric values are:

n	x_{n-1}	x_n	N_n	D_n	$x_{n+1} - x_n$
1	2.000000	1.900000	−0.000740	−0.174005	−0.004253
2	1.900000	1.895747	−0.000002	−0.006986	−0.000252
3	1.895747	1.895494	0		0

$x_3 = 1.895494$ is exact to 6D. See Example 4. ■

Summary of Methods. The methods for computing solutions s of $f(x) = 0$ with given continuous (or differentiable) $f(x)$ start with an initial approximation x_0 of s and generate a sequence $x_1, x_2, \cdots$ by **iteration**. **Fixed-point methods** solve $f(x) = 0$ written as $x = g(x)$, so that s is a *fixed point* of g, that is, $s = g(s)$. For $g(x) = x - f(x)/f'(x)$ this is **Newton's method**, which, for good x_0 and simple zeros, converges quadratically (and for multiple zeros linearly). From Newton's method the **secant method** follows by replacing $f'(x)$ by a difference quotient. The ***bisection method*** and the ***method of false position*** in Problem Set 19.2 always converge, but often slowly.

ROBLEM SET 19.2

11 FIXED-POINT ITERATION

ve by fixed-point iteration and answer related stions where indicated. Show details.

Monotone sequence. Why is the sequence in Example 1 monotone? Why not in Example 2?

Do the iterations (b) in Example 2. Sketch a figure similar to Fig. 427. Explain what happens.

$f = x - 0.5\cos x = 0, \quad x_0 = 1$. Sketch a figure.

$f = x - \operatorname{cosec} x$ the zero near $x = 1$.

Sketch $f(x) = x^3 - 5.00x^2 + 1.01x + 1.88$, showing roots near ± 1 and 5. Write $x = g(x) = (5.00x^2 - 1.01x + 1.88)/x^2$. Find a root by starting from $x_0 = 5, 4, 1, -1$. Explain the (perhaps unexpected) results.

Find a form $x = g(x)$ of $f(x) = 0$ in Prob. 5 that yields convergence to the root near $x = 1$.

Find the smallest positive solution of $\sin x = e^{-x}$.

Elasticity. Solve $x \cosh x = 1$. (Similar equations appear in vibrations of beams; see Problem Set 12.3.)

Drumhead. Bessel functions. A partial sum of the Maclaurin series of $J_0(x)$ (Sec. 5.5) is $f(x) = 1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{1}{2304}x^6$. Conclude from a sketch that $f(x) = 0$ near $x = 2$. Write $f(x) = 0$ as $x = g(x)$ (by dividing $f(x)$ by $\frac{1}{4}x$ and taking the resulting x-term to the other side). Find the zero. (See Sec. 12.10 for the importance of these zeros.)

10. **CAS EXPERIMENT. Convergence.** Let $f(x) = x^3 + 2x^2 - 3x - 4 = 0$. Write this as $x = g(x)$, for g choosing (1) $(x^3 - f)^{1/3}$, (2) $(x^2 - \frac{1}{2}f)^{1/2}$, (3) $x + \frac{1}{3}f$, (4) $(x^3 - f)/x^2$, (5) $(2x^2 - f)/(2x)$, and (6) $x - f/f'$ and in each case $x_0 = 1.5$. Find out about convergence and divergence and the number of steps to reach 6S-values of a root.

11. **Existence of fixed point.** Prove that if g is continuous in a closed interval I and its range lies in I, then the equation $x = g(x)$ has at least one solution in I. Illustrate that it may have more than one solution in I.

12–18 NEWTON'S METHOD

Apply Newton's method (6S-accuracy). First sketch the function(s) to see what is going on.

12. **Cube root.** Design a Newton iteration. Compute $\sqrt[3]{7}, x_0 = 2$.

13. $f = 2x - \cos x, \quad x_0 = 1$. Compare with Prob. 3.

14. What happens in Prob. 13 for any other x_0?

15. **Dependence on x_0.** Solve Prob. 5 by Newton's method with $x_0 = 5, 4, 1, -3$. Explain the result.

16. **Legendre polynomials.** Find the largest root of the Legendre polynomial $P_5(x)$ given by $P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$ (Sec. 5.3) (to be needed in ***Gauss integration*** in Sec. 19.5) **(a)** by Newton's method, **(b)** from a quadratic equation.

17. **Heating, cooling.** At what time x (4S-accuracy only) will the processes governed by $f_1(x) = 100(1 - e^{-0.2x})$ and $f_2(x) = 40e^{-0.01x}$ reach the same temperature? Also find the latter.

18. **Vibrating beam.** Find the solution of $\cos x \cosh x = 1$ near $x = \frac{3}{2}\pi$. (This determines a frequency of a vibrating beam; see Problem Set 12.3.)

19. **Method of False Position (Regula falsi).** Figure 430 shows the idea. We assume that f is continuous. We compute the x-intercept c_0 of the line through $(a_0, f(a_0)), (b_0, f(b_0))$. If $f(c_0) = 0$, we are done. If $f(a_0)f(c_0) < 0$ (as in Fig. 430), we set $a_1 = a_0, b_1 = c_0$ and repeat to get c_1, etc. If $f(a_0)f(c_0) > 0$, then $f(c_0)f(b_0) < 0$ and we set $a_1 = c_0, b_1 = b_0$, etc.
    (a) Algorithm. Show that
    $$c_0 = \frac{a_0 f(b_0) - b_0 f(a_0)}{f(b_0) - f(a_0)}$$
    and write an algorithm for the method.
    (b) Solve $x^4 = 2$, $\cos x = \sqrt{x}$, and $x + \ln x = 2$, with $a = 1, b = 2$.

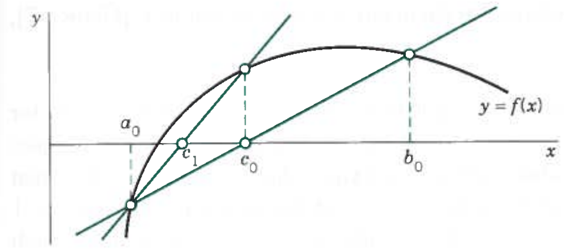

Fig. 430. Method of false position

20. **TEAM PROJECT. Bisection Method.** This simple but slowly convergent method for finding a solution of $f(x) = 0$ with continuous f is based on the **intermediate value theorem**, which states that if a continuous function f has opposite signs at some $x = a$ and $x = b (> a)$, that is, either $f(a) < 0, f(b) > 0$ or $f(a) > 0, f(b) < 0$, then f must be 0 somewhere on $[a, b]$. The solution is found by repeated bisection of the interval and in each iteration picking that half which also satisfies that sign condition.
    (a) Algorithm. Write an algorithm for the method.
    (b) Comparison. Solve $x = \cos x$ by Newton's method and by bisection. Compare.
    (c) Solve $e^{-x} = \ln x$ and $e^x + x^4 + x = 2$ by bisection.

21–22 SECANT METHOD

Solve, using x_0 and x_1 as indicated:

21. $e^{-x} - \tan x = 0, \quad x_0 = 1, \quad x_1 = 0.7$

22. $x = \cos x, \quad x_0 = 0.5, \quad x_1 = 1$

23. **WRITING PROJECT. Solution of Equations.** Compare the methods in this section and problem set, discussing advantages and disadvantages in terms of examples of your own. No proofs, just motivations and ideas.

19.3 Interpolation

We are given the values of a function $f(x)$ at different points $x_0, x_1, \cdots, x_n$. We want to find approximate values of the function $f(x)$ for "new" x's that lie between these points for which the function values are given. This process is called **interpolation**. The student should pay close attention to this section as interpolation forms the underlying foundation for both Secs. 19.4 and 19.5. Indeed, interpolation allows us to develop formulas for numeric integration and differentiation as shown in Sec. 19.5.

Continuing our discussion, we write these given values of a function f in the form

$$f_0 = f(x_0), \qquad f_1 = f(x_1), \qquad \cdots, \qquad f_n = f(x_n)$$

or as ordered pairs

$$(x_0, f_0), \qquad (x_1, f_1), \qquad \cdots, \qquad (x_n, f_n).$$

Where do these given function values come from? They may come from a "mathematical" function, such as a logarithm or a Bessel function. More frequently, they may be measured or automatically recorded values of an "empirical" function, such as air resistance of a