EXAMPLE 6 Shifted Data Problems

This means initial value problems with initial conditions given at some $t = t_0 > 0$ instead of $t = 0$. For such a problem set $t = \tilde{t} + t_0$, so that $t = t_0$ gives $\tilde{t} = 0$ and the Laplace transform can be applied. For instance, solve

$$y'' + y = 2t, \quad y(\tfrac{1}{4}\pi) = \tfrac{1}{2}\pi, \quad y'(\tfrac{1}{4}\pi) = 2 - \sqrt{2}.$$

Solution. We have $t_0 = \frac{1}{4}\pi$ and we set $t = \tilde{t} + \frac{1}{4}\pi$. Then the problem is

$$\tilde{y}'' + \tilde{y} = 2(\tilde{t} + \tfrac{1}{4}\pi), \quad \tilde{y}(0) = \tfrac{1}{2}\pi, \quad \tilde{y}'(0) = 2 - \sqrt{2}$$

where $\tilde{y}(\tilde{t}) = y(t)$. Using (2) and Table 6.1 and denoting the transform of $\tilde{y}$ by $\tilde{Y}$, we see that the subsidiary equation of the "shifted" initial value problem is

$$s^2\tilde{Y} - s \cdot \tfrac{1}{2}\pi - (2 - \sqrt{2}) + \tilde{Y} = \frac{2}{s^2} + \frac{\frac{1}{2}\pi}{s}, \qquad \text{thus} \qquad (s^2 + 1)\tilde{Y} = \frac{2}{s^2} + \frac{\frac{1}{2}\pi}{s} + \frac{1}{2}\pi s + 2 - \sqrt{2}.$$

Solving this algebraically for $\tilde{Y}$, we obtain

$$\tilde{Y} = \frac{2}{(s^2 + 1)s^2} + \frac{\frac{1}{2}\pi}{(s^2 + 1)s} + \frac{\frac{1}{2}\pi s}{s^2 + 1} + \frac{2 - \sqrt{2}}{s^2 + 1}.$$

The inverse of the first two terms can be seen from Example 3 (with $\omega = 1$), and the last two terms give cos and sin,

$$\tilde{y} = \mathcal{L}^{-1}(\tilde{Y}) = 2(\tilde{t} - \sin \tilde{t}) + \tfrac{1}{2}\pi(1 - \cos \tilde{t}) + \tfrac{1}{2}\pi \cos \tilde{t} + (2 - \sqrt{2}) \sin \tilde{t}$$
$$= 2\tilde{t} + \tfrac{1}{2}\pi - \sqrt{2} \sin \tilde{t}.$$

Now $\tilde{t} = t - \frac{1}{4}\pi$, $\sin \tilde{t} = \frac{1}{\sqrt{2}}(\sin t - \cos t)$, so that the answer (the solution) is

$$y = 2t - \sin t + \cos t.$$ ■

PROBLEM SET 6.2

1–11 INITIAL VALUE PROBLEMS (IVPS)

Solve the IVPs by the Laplace transform. If necessary, use partial fraction expansion as in Example 4 of the text. Show all details.

1. $y' + \frac{2}{5}y = -4 \cos 2t, \quad y(0) = 0$
2. $y' + 2y = 0, \quad y(0) = 1.5$
3. $y'' + y' - 6y = 0, \quad y(0) = 1, \quad y'(0) = 1$
4. $y'' + 9y = 10e^{-t}, \quad y(0) = 0, \quad y'(0) = 0$
5. $y'' - \frac{1}{4}y = 0, \quad y(0) = 12, \quad y'(0) = 0$
6. $y'' - 6y' + 5y = 29 \cos 2t, \quad y(0) = 3.2, \quad y'(0) = 6.2$
7. $y'' + 7y' + 12y = 21e^{3t}, \quad y(0) = 3.5, \quad y'(0) = -10$
8. $y'' - 4y' + 4y = 0, \quad y(0) = 8.1, \quad y'(0) = 3.9$
9. $y'' - 3y' + 2y = 4t - 8, \quad y(0) = 2, \quad y'(0) = 7$
10. $y'' + 0.04y = 0.02t^2, \quad y(0) = -25, \quad y'(0) = 0$
11. $y'' + 3y' + 2.25y = 9t^3 + 64, \quad y(0) = 1, \quad y'(0) = 31.5$

12–15 SHIFTED DATA PROBLEMS

Solve the shifted data IVPs by the Laplace transform. Show the details.

12. $y'' + 2y' - 3y = 0, \quad y(2) = -3, \quad y'(2) = -5$
13. $y' - 6y = 0, \quad y(-1) = 4$
14. $y'' + 2y' + 5y = 50t - 100, \quad y(2) = -4, \quad y'(2) = 14$
15. $y'' + 3y' - 4y = 6e^{2t-3}, \quad y(1.5) = 4, \quad y'(1.5) = 5$

16–21 OBTAINING TRANSFORMS BY DIFFERENTIATION

Using (1) or (2), find $\mathcal{L}(f)$ if $f(t)$ equals:

16. $t \cos 4t$
17. te^{at}
18. $\cos^2 2t$
19. $\cos^2 \omega t$
20. $\sin^4 t$. Use Prob. 19.
21. $\sinh^2 t$

22. **PROJECT. Further Results by Differentiation.** Proceeding as in Example 1, obtain

(a) $\mathcal{L}(t \cos \omega t) = \dfrac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$

and from this and Example 1: **(b)** formula 21, **(c)** 22, **(d)** 23 in Sec. 6.9,

(e) $\mathcal{L}(t \cosh at) = \dfrac{s^2 + a^2}{(s^2 - a^2)^2},$

(f) $\mathcal{L}(t \sinh at) = \dfrac{2as}{(s^2 - a^2)^2}.$

23–29 INVERSE TRANSFORMS BY INTEGRATION

Using Theorem 3, find $f(t)$ if $\mathcal{L}(F)$ equals:

23. $\dfrac{2}{s^2 + s/3}$
24. $\dfrac{20}{s^3 - 2\pi s^2}$
25. $\dfrac{1}{s(s^2 + \omega^2/4)}$
26. $\dfrac{1}{s^4 - s^2}$
27. $\dfrac{s + 8}{s^4 + 4s^2}$
28. $\dfrac{3s + 4}{s^4 + k^2s^2}$
29. $\dfrac{1}{s^3 + as^2}$

30. **PROJECT. Comments on Sec.** why Theorems 1 and 2 are n Theorem 3.

(b) Extend Theorem 1 by sho continuous, except for an ordinar jump) at some $t = a$ (>0), the other as in Theorem 1, then (see Fig. 11

(1*) $\mathcal{L}(f') = s\mathcal{L}(f) - f(0) - [f(a +$

(c) Verify (1*) for $f(t) = e^{-t}$ if $t > 1$.

(d) Compare the Laplace transfo with the method in Chap. 2. Gi own to illustrate the advantages o (to the extent we have seen them

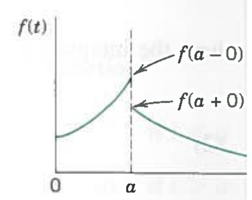

Fig. 117. Formula (

6.3 Unit Step Function (Heaviside Functi Second Shifting Theorem (t-Shifting)

This section and the next one are extremely important because we s point where the Laplace transform method shows its real power in a superiority over the classical approach of Chap. 2. The reason is that two auxiliary functions, the *unit step function* or *Heaviside function u(* *Dirac's delta* $\delta(t - a)$ (in Sec. 6.4). These functions are suitable for s complicated right sides of considerable engineering interest, such as si (driving forces) that are discontinuous or act for some time only, pe general than just cosine and sine, or impulsive forces acting for an insta for example).

Unit Step Function (Heaviside Function) u(t

The **unit step function** or **Heaviside function** $u(t - a)$ is 0 for $t < a$, 1 at $t = a$ (where we can leave it undefined), and is 1 for $t > a$, in a

(1) $$u(t - a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases}$$