Using Theorem 3 in Sec. 6.2 and formula (1) in this section, we obtain the subsidiary equation

$$RI(s) + \frac{I(s)}{sC} = \frac{V_0}{s}[e^{-as} - e^{-bs}].$$

Solving this equation algebraically for $I(s)$, we get

$$I(s) = F(s)(e^{-as} - e^{-bs}) \qquad \text{where} \qquad F(s) = \frac{V_0/R}{s + 1/(RC)} \qquad \text{and} \qquad \mathcal{L}^{-1}(F) = \frac{V_0}{R}e^{-t/(RC)},$$

the last expression being obtained from Table 6.1 in Sec. 6.1. Hence Theorem 1 yields the solution (Fig. 124)

$$i(t) = \mathcal{L}^{-1}(I) = \mathcal{L}^{-1}\{e^{-as}F(s)\} - \mathcal{L}^{-1}\{e^{-bs}F(s)\} = \frac{V_0}{R}[e^{-(t-a)/(RC)}u(t-a) - e^{-(t-b)/(RC)}u(t-b)];$$

that is, $i(t) = 0$ if $t < a$, and

$$i(t) = \begin{cases} K_1 e^{-t/(RC)} & \text{if } a < t < b \\ (K_1 - K_2)e^{-t/(RC)} & \text{if } a > b \end{cases}$$

where $K_1 = V_0 e^{a/(RC)}/R$ and $K_2 = V_0 e^{b/(RC)}/R$. ■

EXAMPLE 4 **Response of an *RLC*-Circuit to a Sinusoidal Input Acting Over a Time Interval**

Find the response (the current) of the *RLC*-circuit in Fig. 125, where $E(t)$ is sinusoidal, acting for a short time interval only, say,

$$E(t) = 100 \sin 400t \quad \text{if } 0 < t < 2\pi \qquad \text{and} \qquad E(t) = 0 \text{ if } t > 2\pi$$

and current and charge are initially zero.

Solution. The electromotive force $E(t)$ can be represented by $(100 \sin 400t)(1 - u(t - 2\pi))$. Hence the model for the current $i(t)$ in the circuit is the integro-differential equation (see Sec. 2.9)

$$0.1i' + 11i + 100\int_0^t i(\tau)\,d\tau = (100 \sin 400t)(1 - u(t - 2\pi)). \qquad i(0) = 0, \quad i'(0) = 0.$$

From Theorems 2 and 3 in Sec. 6.2 we obtain the subsidiary equation for $I(s) = \mathcal{L}(i)$

$$0.1sI + 11I + 100\frac{I}{s} = \frac{100 \cdot 400s}{s^2 + 400^2}\left(\frac{1}{s} - \frac{e^{-2\pi s}}{s}\right).$$

Solving it algebraically and noting that $s^2 + 110s + 1000 = (s + 10)(s + 100)$, we obtain

$$I(s) = \frac{1000 \cdot 400}{(s + 10)(s + 100)}\left(\frac{s}{s^2 + 400^2} - \frac{se^{-2\pi s}}{s^2 + 400^2}\right).$$

For the first term in the parentheses $(\cdots)$ times the factor in front of them we use the partial fraction expansion

$$\frac{400{,}000s}{(s + 10)(s + 100)(s^2 + 400^2)} = \frac{A}{s + 10} + \frac{B}{s + 100} + \frac{Ds + K}{s^2 + 400^2}.$$

Now determine A, B, D, K by your favorite method or by a CAS or as follows. Multiplication by the common denominator gives

$$400{,}000s = A(s + 100)(s^2 + 400^2) + B(s + 10)(s^2 + 400^2) + (Ds + K)(s + 10)(s + 100).$$

We set $s = -10$ and -100 and then equate the sums of the s^3 and s^2 terms to zero, obtaining (all values rounded)

$(s = -10)$	$-4{,}000{,}000 = 90(10^2 + 400^2)A,$	$A = -0.27760$
$(s = -100)$	$-40{,}000{,}000 = -90(100^2 + 400^2)B,$	$B = 2.6144$
(s^3-terms)	$0 = A + B + D,$	$D = -2.3368$
(s^2-terms)	$0 = 100A + 10B + 110D + K,$	$K = 258.66.$

Since $K = 258.66 = 0.6467 \cdot 400$, we thus obtain for the first term I_1 in $I = I_1 - I_2$

$$I_1 = -\frac{0.2776}{s + 10} + \frac{2.6144}{s + 100} - \frac{2.3368s}{s^2 + 400^2} + \frac{0.6467 \cdot 400}{s^2 + 400^2}.$$

From Table 6.1 in Sec. 6.1 we see that its inverse is

$$i_1(t) = -0.2776e^{-10t} + 2.6144e^{-100t} - 2.3368 \cos 400t + 0.6467 \sin 400t.$$

This is the current $i(t)$ when $0 < t < 2\pi$. It agrees for $0 < t < 2\pi$ with that in Example 1 of Sec. 2.9 (except for notation), which concerned the same *RLC*-circuit. Its graph in Fig. 63 in Sec. 2.9 shows that the exponential terms decrease very rapidly. Note that the present amount of work was substantially less.

The second term I_1 of I differs from the first term by the factor $e^{-2\pi s}$. Since $\cos 400(t - 2\pi) = \cos 400t$ and $\sin 400(t - 2\pi) = \sin 400t$, the second shifting theorem (Theorem 1) gives the inverse $i_2(t) = 0$ if $0 < t < 2\pi$, and for $> 2\pi$ it gives

$$i_2(t) = -0.2776e^{-10(t-2\pi)} + 2.6144e^{-100(t-2\pi)} - 2.3368 \cos 400t + 0.6467 \sin 400t.$$

Hence in $i(t)$ the cosine and sine terms cancel, and the current for $t > 2\pi$ is

$$i(t) = -0.2776(e^{-10t} - e^{-10(t-2\pi)}) + 2.6144(e^{-100t} - e^{-100(t-2\pi)}).$$

It goes to zero very rapidly, practically within 0.5 sec. ■

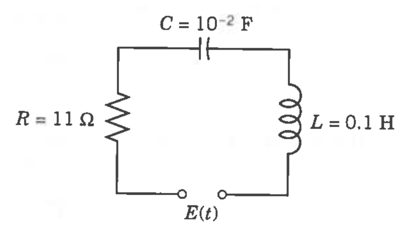

Fig. 125. *RLC*-circuit in Example 4

PROBLEM SET 6.3

1. Report on Shifting Theorems. Explain and compare the different roles of the two shifting theorems, using your own formulations and simple examples. Give no proofs.

2–11 SECOND SHIFTING THEOREM, UNIT STEP FUNCTION

Sketch or graph the given function, which is assumed to be zero outside the given interval. Represent it, using unit step functions. Find its transform. Show the details of your work.

2. $t\ (0 < t < 2)$
3. $t - 3\ (t > 3)$
4. $\cos 2t\ (0 < t < \pi)$
5. $e^{-t}\ (0 < t < \pi)$
6. $\sin \pi t\ (2 < t < 4)$
7. $e^{-\pi/2t}\ (1 < t < 3)$
8. $t^2\ (1 < t < 2)$
9. $2t^2\ (t > \frac{5}{2})$
10. $\sinh t\ (0 < t < 2)$
11. $\sin t\ (\pi/2 < t < \pi)$

12–17 INVERSE TRANSFORMS BY THE 2ND SHIFTING THEOREM

Find and sketch or graph $f(t)$ if $\mathcal{L}(f)$ equals

12. $e^{-2s}/(s - 1)^3$
13. $4(1 - e^{-\pi s})/(s^2 + 4)$
14. $4(e^{-2s} - 2e^{-5s})/s$
15. e^{-2s}/s^6
16. $2(e^{-s} - e^{-3s})/(s^2 - 4)$
17. $(1 + e^{-2\pi(s+1)})(s + 1)/((s + 1)^2 + 1)$