...tegral from 0 to ∞ because $B(w)$ is even (odd times odd

...ted out that the main application of the Fourier integral ...equations. However, these representations also help in ...wing example shows for integrals from 0 to ∞.

...urier sine integrals of $f(x) = e^{-kx}$, where $x > 0$ and $k > 0$ (Fig. 284). ...-called Laplace integrals.

$$\ldots) = \frac{2}{\pi}\int_0^\infty e^{-kv} \cos wv \, dv.$$ Now, by integration by parts,

$$\ldots = -\frac{k}{k^2 + w^2} e^{-kv}\left(-\frac{w}{k}\sin wv + \cos wv\right).$$

...ls $-k/(k^2 + w^2)$. If v approaches infinity, that expression approaches ...hus $2/\pi$ times the integral from 0 to ∞ gives

$$A(w) = \frac{2k/\pi}{k^2 + w^2}.$$

...in (10) we thus obtain the Fourier cosine integral representation

$$\ldots) = e^{-kx} = \frac{2k}{\pi}\int_0^\infty \frac{\cos wx}{k^2 + w^2}\, dw \qquad (x > 0, \quad k > 0).$$

$$\int_0^\infty \frac{\cos wx}{k^2 + w^2}\, dw = \frac{\pi}{2k} e^{-kx} \qquad (x > 0, \quad k > 0).$$

$$\ldots) = \frac{2}{\pi}\int_0^\infty e^{-kv} \sin wv \, dv.$$ By integration by parts,

$$\ldots = -\frac{w}{k^2 + w^2} e^{-kv}\left(\frac{k}{w}\sin wv + \cos wv\right).$$

...approaches 0 as $v \to \infty$. Thus

$$B(w) = \frac{2w/\pi}{k^2 + w^2}.$$

...e integral representation

$$\ldots) = e^{-kx} = \frac{2}{\pi}\int_0^\infty \frac{w \sin wx}{k^2 + w^2}\, dw.$$

$$\int_0^\infty \frac{w \sin wx}{k^2 + w^2}\, dw = \frac{\pi}{2} e^{-kx} \qquad (x > 0, \quad k > 0).$$

Laplace integrals.

PROBLEM SET 11.7

1–6 EVALUATION OF INTEGRALS

Show that the integral represents the indicated function. *Hint.* Use (5), (10), or (11); the integral tells you which one, and its value tells you what function to consider. Show your work in detail.

1. $$\int_0^\infty \frac{\cos xw + w \sin xw}{1 + w^2}\, dx = \begin{cases} 0 & \text{if } x < 0 \\ \pi/2 & \text{if } x = 0 \\ \pi e^{-x} & \text{if } x > 0 \end{cases}$$

2. $$\int_0^\infty \frac{\sin \pi w \sin xw}{1 - w^2}\, dw = \begin{cases} \frac{\pi}{2}\sin x & \text{if } 0 \leqq x \leqq \pi \\ 0 & \text{if } x > \pi \end{cases}$$

3. $$\int_0^\infty \frac{1 - \cos \pi w}{w} \sin xw \, dw = \begin{cases} \frac{1}{2}\pi & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$$

4. $$\int_0^\infty \frac{\cos \frac{1}{2}\pi w}{1 - w^2} \cos xw \, dw = \begin{cases} \frac{1}{2}\pi \cos x & \text{if } 0 < |x| < \frac{1}{2}\pi \\ 0 & \text{if } |x| \geqq \frac{1}{2}\pi \end{cases}$$

5. $$\int_0^\infty \frac{\sin w - w \cos w}{w^2} \sin xw \, dw = \begin{cases} \frac{1}{2}\pi x & \text{if } 0 < x < 1 \\ \frac{1}{4}\pi & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases}$$

6. $$\int_0^\infty \frac{w^3 \sin xw}{w^4 + 4}\, dw = \tfrac{1}{2}\pi e^{-x} \cos x \quad \text{if } x > 0$$

7–12 FOURIER COSINE INTEGRAL REPRESENTATIONS

Represent $f(x)$ as an integral (10).

7. $f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$

8. $f(x) = \begin{cases} x^2 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$

9. $f(x) = 1/(1 + x^2)$ $[x > 0.$ *Hint.* See (13).]

10. $f(x) = \begin{cases} a^2 - x^2 & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

11. $f(x) = \begin{cases} \sin x & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$

12. $f(x) = \begin{cases} e^{-x} & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

13. CAS EXPERIMENT. Approximate Fourier Cosine Integrals. Graph the integrals in Prob. 7, 9, and 11 as functions of x. Graph approximations obtained by replacing ∞ with finite upper limits of your choice. Compare the quality of the approximations. Write a short report on your empirical results and observations.

14. PROJECT. Properties of Fourier Integrals

(a) Fourier cosine integral. Show that (10) implies

$$\text{(a1)} \qquad f(ax) = \frac{1}{a}\int_0^\infty A\left(\frac{w}{a}\right) \cos xw \, dw$$
$$(a > 0) \qquad \textit{(Scale change)}$$

$$\text{(a2)} \qquad xf(x) = \int_0^\infty B^*(w) \sin xw \, dw,$$
$$B^* = -\frac{dA}{dw}, \qquad A \text{ as in (10)}$$

$$\text{(a3)} \qquad x^2 f(x) = \int_0^\infty A^*(w) \cos xw \, dw,$$
$$A^* = -\frac{d^2A}{dw^2}.$$

(b) Solve Prob. 8 by applying (a3) to the result of Prob. 7.

(c) Verify (a2) for $f(x) = 1$ if $0 < x < a$ and $f(x) = 0$ if $x > a$.

(d) Fourier sine integral. Find formulas for the Fourier sine integral similar to those in (a).

15. CAS EXPERIMENT. Sine Integral. Plot Si(u) for positive u. Does the sequence of the maximum and minimum values give the impression that it converges and has the limit $\pi/2$? Investigate the Gibbs phenomenon graphically.

16–20 FOURIER SINE INTEGRAL REPRESENTATIONS

Represent $f(x)$ as an integral (11).

16. $f(x) = \begin{cases} x & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

17. $f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$

18. $f(x) = \begin{cases} \cos x & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$

19. $f(x) = \begin{cases} e^x & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$

20. $f(x) = \begin{cases} e^{-x} & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$