EXAMPLE 6 Newton's Forward and Backward Interpolations

Compute a 7D-value of the Bessel function $J_0(x)$ for $x = 1.72$ from the four values in the following table, using (a) Newton's forward formula (14), (b) Newton's backward formula (18).

j_{for}	j_{back}	x_j	$J_0(x_j)$	1st Diff.	2nd Diff.	3rd Diff.
0	−3	1.7	0.3979849			
				−0.0579985		
1	−2	1.8	0.3399864		−0.0001693	
				−0.0581678		0.0004093
2	−1	1.9	0.2818186		0.0002400	
				−0.0579278		
3	0	2.0	0.2238908			

Solution. The computation of the differences is the same in both cases. Only their notation differs.

(a) Forward. In (14) we have $r = (1.72 - 1.70)/0.1 = 0.2$, and j goes from 0 to 3 (see first column). In each column we need the first given number, and (14) thus gives

$$J_0(1.72) \approx 0.3979849 + 0.2(-0.0579985) + \frac{0.2(-0.8)}{2}(-0.0001693) + \frac{0.2(-0.8)(-1.8)}{6} \cdot 0.0004093$$

$$= 0.3979849 - 0.0115997 + 0.0000135 + 0.0000196 = 0.3864183,$$

which is exact to 6D, the exact 7D-value being 0.3864185.

(b) Backward. For (18) we use j shown in the second column, and in each column the last number. Since $r = (1.72 - 2.00)/0.1 = -2.8$, we thus get from (18)

$$J_0(1.72) \approx 0.2238908 - 2.8(-0.0579278) + \frac{-2.8(-1.8)}{2} \cdot 0.0002400 + \frac{-2.8(-1.8)(-0.8)}{6} \cdot 0.0004093$$

$$= 0.2238908 + 0.1621978 + 0.0006048 - 0.0002750$$

$$= 0.3864184.$$

There is a third notation for differences, called the **central difference notation.** It is used in numerics for ODEs and certain interpolation formulas. See Ref. [E5] listed in App. 1.

PROBLEM SET 19.3

. **Linear interpolation.** Calculate $p_1(x)$ in Example 1 and from it ln 9.3.

. **Error estimate.** Estimate the error in Prob. 1 by (5).

. **Quadratic interpolation. Gamma function.** Calculate the Lagrange polynomial $p_2(x)$ for the values $\Gamma(1.00) = 1.0000$, $\Gamma(1.02) = 0.9888$, $\Gamma(1.04) = 0.9784$ of the gamma function [(24) in App. A3.1] and from it approximations of $\Gamma(1.01)$ and $\Gamma(1.03)$.

. **Error estimate for quadratic interpolation.** Estimate the error for $p_2(9.2)$ in Example 2 from (5).

. **Linear and quadratic interpolation.** Find $e^{-0.25}$ and $e^{-0.75}$ by linear interpolation of e^{-x} with $x_0 = 0$, $x_1 = 0.5$ and $x_0 = 0.5$, $x_1 = 1$, respectively. Then find $p_2(x)$ by quadratic interpolation of e^{-x} with $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$ and from it $e^{-0.25}$ and $e^{-0.75}$. Compare the errors. Use 4S-values of e^{-x}.

6. **Extrapolation.** Does a sketch of the product of the $(x - x_j)$ in (5) for the data in Example 2 indicate that extrapolation is likely to involve larger errors than interpolation does?

7. **Error function** (35) in App. A3.1. Calculate the Lagrange polynomial $p_2(x)$ for the 5S-values $f(0.25) = 0.27633$, $f(0.5) = 0.52050$, $f(1.0) = 0.84270$ and from $p_2(x)$ an approximation of $f(0.75)$ $(= 0.71116)$.