Evaluating this at x_0, x_1, x_2, we obtain the "three-point formulas"

$$(14)\qquad \begin{aligned} &\text{(a)}\quad f_0' \approx \frac{1}{2h}(-3f_0 + 4f_1 - f_2),\\ &\text{(b)}\quad f_1' \approx \frac{1}{2h}(-f_0 + f_2),\\ &\text{(c)}\quad f_2' \approx \frac{1}{2h}(f_0 - 4f_1 + 3f_2). \end{aligned}$$

Applying the same idea to the Lagrange polynomial $p_4(x)$, we obtain similar formulas, in particular,

$$(15)\qquad f_2' \approx \frac{1}{12h}(f_0 - 8f_1 + 8f_3 - f_4).$$

Some examples and further formulas are included in the problem set as well as in Ref. [E5] listed in App. 1.

PROBLEM SET 19.5

1–4 RECTANGULAR AND TRAPEZOIDAL RULES

1. **Rectangular rule.** Evaluate the integral in Example 1 by the rectangular rule (1) with subintervals of length 0.1. Compare with Example 1. (6S-exact: 0.746824)
2. **Bounds for (1).** Derive a formula for lower and upper bounds for the rectangular rule. Apply it to Prob. 1.
3. **Error estimation by halfing.** Integrate $f(x) = x^4$ from 0 to 1 by (2) with $h = 1, h = 0.5, h = 0.25$ and estimate the error for $h = 0.5$ and $h = 0.25$ by (5).
4. **Stability.** Prove that the trapezoidal rule is stable with respect to rounding.

5–8 SIMPSON'S RULE

Evaluate the integrals $A = \int_1^2 \frac{dx}{x}$, $B = \int_0^{0.4} xe^{-x^2}\,dx$, $J = \int_0^1 \frac{dx}{1+x^2}$ by Simpson's rule with $2m$ as indicated, and compare with the exact value known from calculus.

5. $A, 2m = 4$
6. $B, 2m = 4$
7. $J, 2m = 4$
8. **Error estimate.** Compute the integral J by Simpson's rule with $2m = 8$ and use the value and that in Prob. 7 to estimate the error by (10).

9–10 NONELEMENTARY INTEGRALS

The following integrals cannot be evaluated by the usual methods of calculus. Evaluate them as indicated. Compare your value with that possibly given by your CAS. Si(x) is the sine integral. $S(x)$ and $C(x)$ are the Fresnel integrals. See App. A3.1. They occur in optics.

$$\mathrm{Si}(x) = \int_0^x \frac{\sin x^*}{x^*}\,dx^*,$$

$$S(x) = \int_0^x \sin(x^{*2})\,dx^*, \quad C(x) = \int_0^x \cos(x^{*2})\,dx^*$$

9. Si(1) by (2), $n = 5, n = 10$, and apply (5).
10. Si(1) by (7), $2m = 2, 2m = 4$

11–12 GAUSS INTEGRATION

Integrate by (11) with $n = 5$:

11. $\cos x$ from 0 to $\frac{1}{2}\pi$
12. $\exp(-x^2)$ from 0 to 1
13. **TEAM PROJECT. Romberg Integration** (W. Romberg, *Norske Videnskab. Trondheim, Fφrh.* 28, Nr. 7, 1955). This method uses the trapezoidal rule and gains precision stepwise by halving h and adding an error estimate. Do this for the integral of $f(x) = e^{-x}$ from $x = 0$ to $x = 2$ with TOL $= 10^{-3}$, as follows.