This method converges for every choice of $\mathbf{x}^{(0)}$ if and only if the spectral radius of $\mathbf{I} - \mathbf{A}$ is less than 1. It has recently gained greater practical interest since on parallel processors all n equations can be solved simultaneously at each iteration step.

For Jacobi, see Sec. 10.3. For exercises, see the problem set.

PROBLEM SET 20.3

1. Verify the solution in Example 1 of the text.
2. Show that for the system in Example 2 the Jacobi iteration diverges. *Hint.* Use eigenvalues.
3. Verify the claim at the end of Example 2.

4–10 GAUSS–SEIDEL ITERATION

Do 5 steps, starting from $\mathbf{x}_0 = [1 \quad 1 \quad 1]^T$ and using 6S in the computation. *Hint.* Make sure that you solve each equation for the variable that has the largest coefficient (why?). Show the details.

4. $$\begin{aligned} 4x_1 - x_2 \phantom{{}+ x_3} &= 21 \\ -x_1 + 4x_2 - x_3 &= -45 \\ - x_2 + 4x_3 &= 33 \end{aligned}$$

5. $$\begin{aligned} 10x_1 + x_2 + x_3 &= 6 \\ x_1 + 10x_2 + x_3 &= 6 \\ x_1 + x_2 + 10x_3 &= 6 \end{aligned}$$

6. $$\begin{aligned} x_2 + 7x_3 &= 25.5 \\ 5x_1 + x_2 &= 0 \\ x_1 + 6x_2 + x_3 &= -10.5 \end{aligned}$$

7. $$\begin{aligned} 5x_1 - 2x_2 &= 18 \\ -2x_1 + 10x_2 - 2x_3 &= -60 \\ - 2x_2 + 15x_3 &= 128 \end{aligned}$$

8. $$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 7 \\ x_1 + 3x_2 + 2x_3 &= 4 \\ 2x_1 + x_2 + 3x_3 &= 7 \end{aligned}$$

9. $$\begin{aligned} 5x_1 + x_2 + 2x_3 &= 19 \\ x_1 + 4x_2 - 2x_3 &= -2 \\ 2x_1 + 3x_2 + 8x_3 &= 39 \end{aligned}$$

10. $$\begin{aligned} 4x_1 + 5x_3 &= 12.5 \\ x_1 + 6x_2 + 2x_3 &= 18.5 \\ 8x_1 + 2x_2 + x_3 &= -11.5 \end{aligned}$$

11. Apply the Gauss–Seidel iteration (3 steps) to the system in Prob. 5, starting from **(a)** 0, 0, 0 **(b)** 10, 10, 10. Compare and comment.

12. In Prob. 5, compute $\mathbf{C}$ **(a)** if you solve the first equation for x_1, the second for x_2, the third for x_3, proving convergence; **(b)** if you nonsensically solve the third equation for x_1, the first for x_2, the second for x_3, proving divergence.

13. **CAS Experiment. Gauss–Seidel Iteration.** **(a)** Write a program for Gauss–Seidel iteration.

(b) Apply the program $\mathbf{A}(t)\mathbf{x} = \mathbf{b}$, to starting from $[0 \quad 0 \quad 0]^T$, where

$$\mathbf{A}(t) = \begin{bmatrix} 1 & t & t \\ t & 1 & t \\ t & t & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

For $t = 0.2, 0.5, 0.8, 0.9$ determine the number of steps to obtain the exact solution to 6S and the corresponding spectral radius of $\mathbf{C}$. Graph the number of steps and the spectral radius as functions of t and comment.

(c) Successive overrelaxation (SOR). Show that by adding and subtracting $\mathbf{x}^{(m)}$ on the right, formula (6) can be written

$$\mathbf{x}^{(m+1)} = \mathbf{x}^{(m)} + \mathbf{b} - \mathbf{L}\mathbf{x}^{(m+1)} - (\mathbf{U} + \mathbf{I})\mathbf{x}^{(m)} \qquad (a_{jj} = 1).$$

Anticipation of further corrections motivates the introduction of an **overrelaxation factor** $\omega > 1$ to get the **SOR formula for Gauss–Seidel**

$$\begin{aligned} \mathbf{x}^{(m+1)} = \mathbf{x}^{(m)} &+ \omega(\mathbf{b} - \mathbf{L}\mathbf{x}^{(m+1)} \\ &- (\mathbf{U} + \mathbf{I})\mathbf{x}^{(m)}) \qquad (a_{jj} = 1) \end{aligned} \tag{14}$$

intended to give more rapid convergence. A recommended value is $\omega = 2/(1 + \sqrt{1 - \rho})$, where ρ is the spectral radius of $\mathbf{C}$ in (7). Apply SOR to the matrix in (b) for $t = 0.5$ and 0.8 and notice the improvement of convergence. (Spectacular gains are made with larger systems.)