Backward Euler Method. Stiff ODEs

The **backward Euler formula** for numerically solving (1) is

$$y_{n+1} = y_n + hf(x_{n+1}, y_{n+1}) \qquad (n = 0, 1, \cdots). \tag{16}$$

This formula is obtained by evaluating the right side at the *new* location (x_{n+1}, y_{n+1}); this is called the **backward Euler scheme**. For known y_n it gives y_{n+1} *implicitly,* so it defines an **implicit method**, in contrast to the Euler method (3), which gives y_{n+1} explicitly. Hence (16) must be solved for y_{n+1}. How difficult this is depends on f in (1). For a linear ODE this provides no problem, as Example 4 (below) illustrates. The method is particularly useful for "stiff" ODEs, as they occur quite frequently in the study of vibrations, electric circuits, chemical reactions, etc. The situation of stiffness is roughly as follows; for details, see, for example, [E5], [E25], [E26] in App. 1.

Error terms of the methods considered so far involve a higher derivative. And we ask what happens if we let h *increase.* Now if the error (the derivative) grows fast but the desired solution also grows fast, nothing will happen. However, if that solution does not grow fast, then with growing h the error term can take over to an extent that the numeric result becomes completely nonsensical, as in Fig. 451. Such an ODE for which h must thus be restricted to small values, and the physical system the ODE models, are called **stiff**. This term is suggested by a mass–spring system with a stiff spring (spring with a large k; see Sec. 2.4). Example 4 illustrates that implicit methods remove the difficulty of increasing h in the case of stiffness: It can be shown that in the application of an implicit method the solution remains stable under any increase of h, although the accuracy decreases with increasing h.

EXAMPLE 4 Backward Euler Method. Stiff ODE

The initial value problem

$$y' = f(x, y) = -20hy + 20x^2 + 2x, \quad y(0) = 1$$

has the solution (verify!)

$$y = e^{-20x} + x^2.$$

The backward Euler formula (16) is

$$y_{n+1} = y_n + hf(x_{n+1}, y_{n+1}) = y_n + h(-20y_{n+1} + 20x_{n+1}^2 + 2x_{n+1}).$$

Noting that $x_{n+1} = x_n + h$, taking the term $-20y_{n+1}$ to the left, and dividing, we obtain

$$y_{n+1} = \frac{y_n + h[20(x_n + h)^2 + 2(x_n + h)]}{1 + 20h}. \tag{16*}$$

The numeric results in Table 21.8 show the following.

Stability of the backward Euler method for $h = 0.05$ and also for $h = 0.2$ with an error increase by about a factor 4 for $h = 0.2$,

Stability of the Euler method for $h = 0.05$ but instability for $h = 0.1$ (Fig. 451),

Stability of RK for $h = 0.1$ but instability for $h = 0.2$.

This illustrates that the ODE is stiff. Note that even in the case of stability the approximation of the solution near $x = 0$ is poor. ■

Stiffness will be considered further in Sec. 21.3 in connection with systems of ODEs.

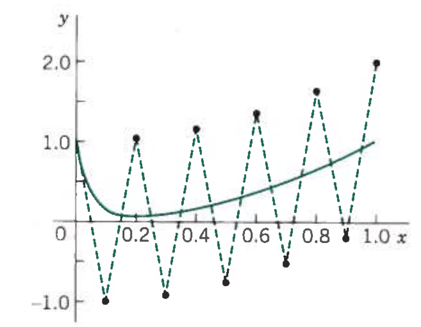

Fig. 451. Euler method with $h = 0.1$ for the stiff ODE in Example 4 and exact solution

Table 21.8 Backward Euler Method (BEM) for Example 6. Comparison with Euler and RK

x	BEM $h = 0.05$	BEM $h = 0.2$	Euler $h = 0.05$	Euler $h = 0.1$	RK $h = 0.1$	RK $h = 0.2$	Exact
0.0	1.00000	1.00000	1.00000	1.00000	1.00000	1.000	1.00000
0.1	0.26188		0.00750	−1.00000	0.34500		0.14534
0.2	0.10484	0.24800	0.03750	1.04000	0.15333	5.093	0.05832
0.3	0.10809		0.08750	−0.92000	0.12944		0.09248
0.4	0.16640	0.20960	0.15750	1.16000	0.17482	25.48	0.16034
0.5	0.25347		0.24750	−0.76000	0.25660		0.25004
0.6	0.36274	0.37792	0.35750	1.36000	0.36387	127.0	0.36001
0.7	0.49256		0.48750	−0.52000	0.49296		0.49001
0.8	0.64252	0.65158	0.63750	1.64000	0.64265	634.0	0.64000
0.9	0.81250		0.80750	−0.20000	0.81255		0.81000
1.0	1.00250	1.01032	0.99750	2.00000	1.00252	3168	1.00000

PROBLEM SET 21.1

1–4 EULER METHOD

Do 10 steps. Solve exactly. Compute the error. Show details.

1. $y' + 0.2y = 0, \quad y(0) = 5, \quad h = 0.2$
2. $y' = \frac{1}{2}\pi\sqrt{1 - y^2}, \quad y(0) = 0, \quad h = 0.1$
3. $y' = (y - x)^2, \quad y(0) = 0, \quad h = 0.1$
4. $y' = (y + x)^2, \quad y(0) = 0, \quad h = 0.1$

5–10 IMPROVED EULER METHOD

Do 10 steps. Solve exactly. Compute the error. Show details.

5. $y' = y, \quad y(0) = 1, \quad h = 0.1$
6. $y' = 2(1 + y^2), \quad y(0) = 0, \quad h = 0.05$
7. $y' - xy^2 = 0, \quad y(0) = 1, \quad h = 0.1$
8. **Logistic population model.** $y' = y - y^2, \quad y(0) = 0.2, \quad h = 0.1$
9. Do Prob. 7 using Euler's method with $h = 0.1$ and compare the accuracy.
10. Do Prob. 7 using the improved Euler method, 20 steps with $h = 0.05$. Compare.

11–17 CLASSICAL RUNGE–KUTTA METHOD OF FOURTH ORDER

Do 10 steps. Compare as indicated. Show details.

11. $y' - xy^2 = 0, \quad y(0) = 1, \quad h = 0.1.$ Compare with Prob. 7. Apply the error estimate (10) to y_{10}.
12. $y' = y - y^2, \quad y(0) = 0.2, \quad h = 0.1.$ Compare with Prob. 8.
13. $y' = 1 + y^2, \quad y(0) = 0, \quad h = 0.1$
14. $y' = (1 - x^{-1})y, \quad y(1) = 1, \quad h = 0.1$
15. $y' + y \tan x = \sin 2x, \quad y(0) = 1, \quad h = 0.1$
16. Do Prob. 15 with $h = 0.2$, 5 steps, and compare the errors with those in Prob. 15.