$$\mathbf{y}_{n+1} = \begin{bmatrix} y_{1,n+1} \\ y_{2,n+1} \end{bmatrix} = \begin{bmatrix} y_{1,n} \\ y_{2,n} \end{bmatrix} + h \begin{bmatrix} y_{2,n+1} \\ -10y_{1,n+1} - 11y_{2,n+1} + 10x_{n+1} + 11 \end{bmatrix}.$$

Reordering terms gives the linear system in the unknowns $y_{1,n+1}$ and $y_{2,n+1}$

$$\begin{aligned} y_{1,n+1} - \qquad\qquad hy_{2,n+1} &= y_{1,n} \\ 10hy_{1,n+1} + (1 + 11h)y_{2,n+1} &= y_{2,n} + 10h(x_n + h) + 11h. \end{aligned}$$

The coefficient determinant is $D = 1 + 11h + 10h^2$, and Cramer's rule (in Sec. 7.6) gives the solution

$$\mathbf{y}_{n+1} = \frac{1}{D}\begin{bmatrix} (1 + 11h)y_{1,n} + hy_{2,n} + 10h^2x_n + 11h^2 + 10h^3 \\ -10hy_{1,n} + y_{2,n} + 10hx_n + 11h + 10h^2 \end{bmatrix}.$$

Table 21.13 Backward Euler Method (BEM) for Example 4. Comparison with Euler and RK

x	BEM $h = 0.2$	BEM $h = 0.4$	Euler $h = 0.1$	Euler $h = 0.2$	RK $h = 0.2$	RK $h = 0.3$	Exact
0.0	2.00000	2.00000	2.00000	2.00000	2.00000	2.00000	2.00000
0.2	1.36667		1.01000	0.00000	1.35207		1.15407
0.4	1.20556	1.31429	1.56100	2.04000	1.18144		1.08864
0.6	1.21574		1.13144	0.11200	1.18585	3.03947	1.15129
0.8	1.29460	1.35020	1.23047	2.20960	1.26168		1.24966
1.0	1.40599		1.34868	0.32768	1.37200		1.36792
1.2	1.53627	1.57243	1.48243	2.46214	1.50257	5.07569	1.50120
1.4	1.67954		1.62877	0.60972	1.64706		1.64660
1.6	1.83272	1.86191	1.78530	2.76777	1.80205		1.80190
1.8	1.99386		1.95009	0.93422	1.96535	8.72329	1.96530
2.0	2.16152	2.18625	2.12158	3.10737	2.13536		2.13534

Table 21.13 shows the following.

Stability of the backward Euler method for $h = 0.2$ and 0.4 (and in fact for any h; try $h = 5.0$) with decreasing accuracy for increasing h

Stability of the Euler method for $h = 0.1$ but instability for $h = 0.2$

Stability of RK for $h = 0.2$ but instability for $h = 0.3$

Figure 452 shows the Euler method for $h = 0.18$, an interesting case with initial jumping (for about $x > 3$) but later monotone following the solution curve of $y = y_1$. See also CAS Experiment 15. ■

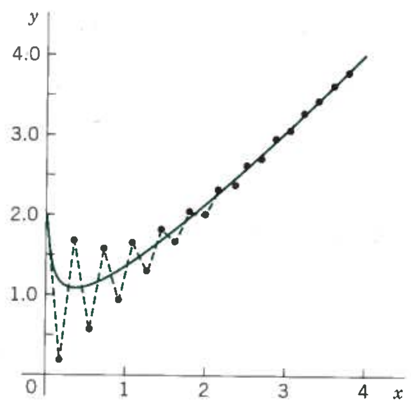

Fig. 452. Euler method with $h = 0.18$ in Example 4

PROBL

1–6 EULER FO... AND SECOND-ORDER ODEs

Solve by the Euler's method. Graph the solution in the y_1y_2-plane. Calculate the errors.

1. $y_1' = 2y_1 - 4y_2$, $y_2' = y_1 - 3y_2$, $y_1(0) = 3$, $y_2(0) = 0$, $h = 0.1$, 10 steps
2. **Spiral.** $y_1' = -y_1 + y_2$, $y_2' = -y_1 - y_2$, $y_1(0) = 0$, $y_2(0) = 4$, $h = 0.2$, 5 steps
3. $y'' + \frac{1}{4}y = 0$, $y(0) = 1$, $y'(0) = 0$, $h = 0.2$, 5 steps
4. $y_1' = -3y_1 + y_2$, $y_2' = y_1 - 3y_2$, $y_1(0) = 2$, $y_2(0) = 0$, $h = 0.1$, 5 steps
5. $y'' - y = x$, $y(0) = 1$, $y'(0) = -2$, $h = 0.1$, 5 steps
6. $y_1' = y_1$, $y_2' = -y_2$, $y_1(0) = 2$, $y_2(0) = 2$, $h = 0.1$, 10 steps

7–10 RK FOR SYSTEMS

Solve by the classical RK.

7. The ODE in Prob. 5. By what factor did the error decrease?
8. The system in Prob. 2
9. The system in Prob. 1
10. The system in Prob. 4
11. **Pendulum equation** $y'' + \sin y = 0$, $y(\pi) = 0$, $y'(\pi) = 1$, as a system, $h = 0.2$, 20 steps. How does your result fit into Fig. 93 in Sec. 4.5?
12. **Bessel Function J_0.** $xy'' + y' + xy = 0$, $y(1) = 0.765198$, $y'(1) = -0.440051$, $h = 0.5$, 5 steps. (This gives the standard solution $J_0(x)$ in Fig. 110 in Sec. 5.4.)
13. Verify the formulas and calculatior equation in Example 2 of the text.
14. **RKN.** The classical RK for a first-orc to second-order ODEs (E. J. Nystı No 13, 1925). If the ODE is y containing y', then

$k_1 = \frac{1}{2}hf(x_n, y_n)$

$k_2 = \frac{1}{2}hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}h(y_n' +$

$k_4 = \frac{1}{2}hf(x_n + h, y_n + h(y_n' + k$

$y_{n+1} = y_n + h(y_n' + \frac{1}{3}(k_1 + 2k_2))$

$y_{n+1}' = y_n' + \frac{1}{3}(k_1 + 4k_2 + k_4).$

Apply this RKN (Runge–Kutta–Nys the Airy ODE in Example 2 with h = obtain approximate values of Ai(x).

15. **CAS EXPERIMENT. Backwa Stiffness.** Extend Example 3 as follc

(a) Verify the values in Table 21.1 graphically as in Fig. 452.

(b) Compute and graph Euler valu "critical" $h = 0.18$ to determine m instability starts.

(c) Compute and graph RK value between 0.2 and 0.3 to find h fc approximation begins to increase aw solution.

(d) Compute and graph backward large h; confirm stability and inv increase for growing h.

21.4 Methods for Elliptic PDEs

We have arrived at the second half of this chapter, which is devoted partial differential equations (PDEs). As we have seen in Chap.12, applications to PDEs, such as in dynamics, elasticity, heat transfer, theory, quantum mechanics, and others. Selected because of thei applications, the PDEs covered here include the Laplace equation, the l the heat equation, and the wave equation. By covering these equatio importance in applications we also selected equations that are importa considerations. Indeed, these equations serve as models for elliptic hyperbolic PDEs. For example, the Laplace equation is a representati elliptic type of PDE, and so forth.