# Butcher tables

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In the lectures we have made use of *Butcher tables* to succinctly summarize Runge–Kutta methods. These notes serve as a quick introduction for those who have not attended lectures, since Kreyszig does not cover the topic.

## 1 Butcher tables specify Runge–Kutta methods

Runge–Kutta methods are numerical methods for solving first-order ordinary differential equations of the form

$$y'(x) = f(x, y(x)) \tag{1}$$

(or systems of such equations).

A Butcher table has the form

$$\begin{array}{c|cccc} c_1 & a_{1,1} & a_{1,2} & \cdots & a_{1,s} \\ c_2 & a_{2,1} & a_{2,2} & \cdots & a_{2,s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s,1} & a_{s,2} & \cdots & a_{s,s} \\ \hline & b_1 & b_2 & \cdots & b_s \end{array}$$

and is a simple mnemonic device for specifying a Runge–Kutta method

$$y_{n+1} = y_n + \sum_{i=1}^{s} b_i k_i$$

where, for $1 \le i \le s$,

$$k_i = hf(x_n + c_i h, y_n + \sum_{j=1}^{s} a_{i,j} k_j).$$

One often also wants to demand that $c_i = \sum_{j=1}^{s} a_{i,j}$, which is satisfied for the four methods we consider.

Observe that the *explicit methods* are precisely those for which the only non-zero entries in the $a_{i,j}$-part of the table lie *strictly below* the diagonal. Entries at or above the diagonal will cause the right hand side of equation (1) to involve $y_{n+1}$, and so give an *implicit method* (verify this for yourself).

### 1.1 Butcher tables for the methods in the course

The four Runge–Kutta methods we have covered have Butcher tables as follows. Verify for yourself that the tables give formulas in agreement with Kreyszig!

- Euler's method (explicit):

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}$$

- Improved Euler's method (Heun's method) (explicit):

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array}$$

- RK4 (explicit):

$$\begin{array}{c|cccc}
0 & 0 & 0 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 & 0 \\
1/2 & 0 & 1/2 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
\hline
 & 1/6 & 1/3 & 1/3 & 1/6
\end{array}$$

- Backwards Euler's method (implicit):

$$\begin{array}{c|c}
1 & 1 \\
\hline
 & 1
\end{array}$$