If $t > 2$, we have to integrate from $\tau = 1$ to 2 (not to t). This gives

$$y(t) = e^{-(t-2)} - \tfrac{1}{2}e^{-2(t-2)} - (e^{-(t-1)} - \tfrac{1}{2}e^{-2(t-1)}).$$

Figure 143 shows the input (the square wave) and the interesting output, which is zero from 0 to 1, then increases, reaches a maximum (near 2.6) after the input has become zero (why?), and finally decreases to zero in a monotone fashion. ■

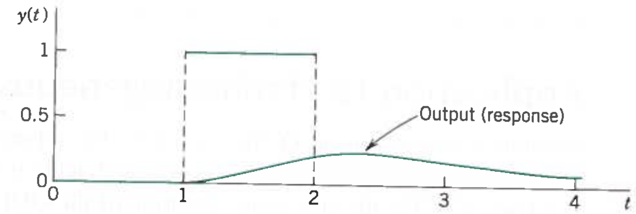

Fig. 143. Square wave and response in Example 5

Integral Equations

Convolution also helps in solving certain **integral equations**, that is, equations in which the unknown function $y(t)$ appears in an integral (and perhaps also outside of it). This concerns equations with an integral of the form of a convolution. Hence these are special and it suffices to explain the idea in terms of two examples and add a few problems in the problem set.

EXAMPLE 6 A Volterra Integral Equation of the Second Kind

Solve the Volterra integral equation of the second kind[3]

$$y(t) - \int_0^t y(\tau)\sin(t-\tau)\,d\tau = t.$$

Solution. From (1) we see that the given equation can be written as a convolution, $y - y * \sin t = t$. Writing $Y = \mathcal{L}(y)$ and applying the convolution theorem, we obtain

$$Y(s) - Y(s)\frac{1}{s^2+1} = Y(s)\frac{s^2}{s^2+1} = \frac{1}{s^2}.$$

The solution is

$$Y(s) = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4} \qquad \text{and gives the answer} \qquad y(t) = t + \frac{t^3}{6}.$$

Check the result by a CAS or by substitution and repeated integration by parts (which will need patience). ■

EXAMPLE 7 Another Volterra Integral Equation of the Second Kind

Solve the Volterra integral equation

$$y(t) - \int_0^t (1+\tau)y(t-\tau)\,d\tau = 1 - \sinh t.$$

[3]If the upper limit of integration is *variable*, the equation is named after the Italian mathematician VITO VOLTERRA (1860–1940), and if that limit is *constant*, the equation is named after the Swedish mathematician ERIK IVAR FREDHOLM (1866–1927). "Of the second kind (first kind)" indicates that y occurs (does not occur) outside of the integral.

Solution. By (1) we can write $y - (1+t) * y = 1 - \sinh t$. Writing $Y = \mathcal{L}(y)$, we obtain by using the convolution theorem and then taking common denominators

$$Y(s)\left[1 - \left(\frac{1}{s} + \frac{1}{s^2}\right)\right] = \frac{1}{s} - \frac{1}{s^2-1}, \qquad \text{hence} \qquad Y(s)\cdot\frac{s^2-s-1}{s^2} = \frac{s^2-1-s}{s(s^2-1)}.$$

$(s^2 - s - 1)/s$ cancels on both sides, so that solving for Y simply gives

$$Y(s) = \frac{s}{s^2-1} \qquad \text{and the solution is} \qquad y(t) = \cosh t.$$ ■

PROBLEM SET 6.5

1–7 CONVOLUTIONS BY INTEGRATION

Find:

1. $1 * (-1)$
2. $1 * \sin \omega t$
3. $e^{-t} * e^t$
4. $(\cos \omega t) * (\cos \omega t)$
5. $(\cos \omega t) * 1$
6. $e^{at} * e^{bt}\ (a \neq b)$
7. $t * e^{-t}$

8–14 INTEGRAL EQUATIONS

Solve by the Laplace transform, showing the details:

8. $y(t) + 4\int_0^t y(\tau)(t-\tau)\,d\tau = 2t$
9. $y(t) + \int_0^t y(\tau)\,d\tau = 2$
10. $y(t) - \int_0^t y(\tau)\sin 2(t-\tau)\,d\tau = \sin 2t$
11. $y(t) - \int_0^t (t-\tau)y(\tau)\,d\tau = 1$
12. $y(t) + \int_0^t y(\tau)\cosh(t-\tau)\,d\tau = t + e^t$
13. $y(t) + 2e^t\int_0^t y(\tau)e^{-\tau}\,d\tau = te^t$
14. $y(t) - \int_0^t y(\tau)(t-\tau)\,d\tau = 2 - \frac{1}{2}t^2$

15. **CAS EXPERIMENT. Variation of a Parameter.**
(a) Replace 2 in Prob. 13 by a parameter k and investigate graphically how the solution curve changes if you vary k, in particular near $k = -2$.
(b) Make similar experiments with an integral equation of your choice whose solution is oscillating.

16. **TEAM PROJECT. Properties of Convolution.** Prove:
(a) Commutativity, $f * g = g * f$
(b) Associativity, $(f * g) * v = f * (g * v)$
(c) Distributivity, $f * (g_1 + g_2) = f * g_1 + f * g_2$
(d) **Dirac's delta.** Derive the sifting formula (4) in Sec. 6.4 by using f_k with $a = 0$ [(1), Sec. 6.4] and applying the mean value theorem for integrals.
(e) **Unspecified driving force.** Show that forced vibrations governed by

$$y'' + \omega^2 y = r(t), \quad y(0) = K_1, \quad y'(0) = K_2$$

with $\omega \neq 0$ and an unspecified driving force $r(t)$ can be written in convolution form,

$$y = \frac{1}{\omega}\sin \omega t * r(t) + K_1 \cos \omega t + \frac{K_2}{\omega}\sin \omega t.$$

17–26 INVERSE TRANSFORMS BY CONVOLUTION

Showing details, find $f(t)$ if $\mathcal{L}(f)$ equals:

17. $\dfrac{5.5}{(s+1.5)(s-4)}$
18. $\dfrac{1}{(s-a)^2}$
19. $\dfrac{2\pi s}{(s^2+\pi^2)^2}$
20. $\dfrac{9}{s(s+3)}$
21. $\dfrac{\omega}{s^2(s^2-\omega^2)}$
22. $\dfrac{e^{-as}}{s(s-2)}$
23. $\dfrac{40.5}{s(s^2-9)}$
24. $\dfrac{240}{(s^2+1)(s^2+25)}$
25. $\dfrac{18s}{(s^2+36)^2}$
26. **Partial Fractions.** Solve Probs. 17, 21, and 23 by partial fraction reduction.