

This shows that the expressions in the parentheses must be the Fourier coefficients b_n of $f(x)$; that is, by (4) in Sec. 11.3,

$$b_n = A_n^* \sinh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a} dx.$$

From this and (16) we see that the solution of our problem is

$$(17) \quad u(x, y) = \sum_{n=1}^{\infty} u_n(x, y) = \sum_{n=1}^{\infty} A_n^* \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}$$

where

$$(18) \quad A_n^* = \frac{2}{a \sinh(n\pi b/a)} \int_0^a f(x) \sin \frac{n\pi x}{a} dx.$$

We have obtained this solution formally, neither considering convergence nor showing that the series for u , u_{xx} , and u_{yy} have the right sums. This can be proved if one assumes that f and f' are continuous and f'' is piecewise continuous on the interval $0 \leq x \leq a$. The proof is somewhat involved and relies on uniform convergence. It can be found in [C4] listed in App. 1.

Unifying Power of Methods. Electrostatics, Elasticity

The Laplace equation (14) also governs the electrostatic potential of electrical charges in any region that is free of these charges. Thus our steady-state heat problem can also be interpreted as an electrostatic potential problem. Then (17), (18) is the potential in the rectangle R when the upper side of R is at potential $f(x)$ and the other three sides are grounded.

Actually, in the steady-state case, the two-dimensional wave equation (to be considered in Secs. 12.8, 12.9) also reduces to (14). Then (17), (18) is the displacement of a rectangular elastic membrane (rubber sheet, drumhead) that is fixed along its boundary, with three sides lying in the xy -plane and the fourth side given the displacement $f(x)$.

This is another impressive demonstration of the *unifying power* of mathematics. It illustrates that *entirely different physical systems may have the same mathematical model* and can thus be treated by the same mathematical methods.

PROBLEM SET 12.6

1. **Decay.** How does the rate of decay of (8) with fixed n depend on the specific heat, the density, and the thermal conductivity of the material?
 $L = \pi$ for $t = 0, 0.1, 0.2, \dots, 1.0$.
2. **Decay.** If the first eigenfunction (8) of the bar decreases to half its value within 20 sec, what is the value of the diffusivity?
3. **Eigenfunctions.** Sketch or graph and compare the first three eigenfunctions (8) with $B_n = 1, c = 1$, and $L = \pi$ for $t = 0, 0.1, 0.2, \dots, 1.0$.
4. **WRITING PROJECT. Wave and Heat Equations.** Compare these PDEs with respect to general behavior of eigenfunctions and kind of boundary and initial

conditions. State the difference between Fig. 291 in Sec. 12.3 and Fig. 295.

5-7 LATERALLY INSULATED BAR

Find the temperature $u(x, t)$ in a bar of silver of length 10 cm and constant cross section of area 1 cm^2 (density 10.6 g/cm^3 , thermal conductivity $1.04 \text{ cal/(cm sec } ^\circ\text{C)}$, specific heat $0.056 \text{ cal/(g } ^\circ\text{C)}$) that is perfectly insulated laterally, with ends kept at temperature 0°C and initial temperature $f(x)^\circ\text{C}$, where

$$5. f(x) = \sin 0.1\pi x$$

$$6. f(x) = 4 - 0.8|x - 5|$$

$$7. f(x) = x(10 - x)$$

8. **Arbitrary temperatures at ends.** If the ends $x = 0$ and $x = L$ of the bar in the text are kept at constant temperatures U_1 and U_2 , respectively, what is the temperature $u_1(x)$ in the bar after a long time (theoretically, as $t \rightarrow \infty$)? First guess, then calculate.

9. In Prob. 8 find the temperature at any time.

10. **Change of end temperatures.** Assume that the ends of the bar in Probs. 5-7 have been kept at 100°C for a long time. Then at some instant, call it $t = 0$, the temperature at $x = L$ is suddenly changed to 0°C and kept at 0°C , whereas the temperature at $x = 0$ is kept at 100°C . Find the temperature in the middle of the bar at $t = 1, 2, 3, 10, 50$ sec. First guess, then calculate.

BAR UNDER ADIABATIC CONDITIONS

"Adiabatic" means no heat exchange with the neighborhood, because the bar is completely insulated, also at the ends. *Physical Information:* The heat flux at the ends is proportional to the value of $\partial u/\partial x$ there.

11. Show that for the completely insulated bar, $u_x(0, t) = 0$, $u_x(L, t) = 0$, $u(x, t) = f(x)$ and separation of variables gives the following solution, with A_n given by (2) in Sec. 11.3.

$$u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} e^{-(cn\pi/L)^2 t}$$

12-15 Find the temperature in Prob. 11 with $L = \pi$, $c = 1$, and

$$12. f(x) = x$$

$$13. f(x) = 1$$

$$14. f(x) = \cos 2x$$

$$15. f(x) = 1 - x/\pi$$

16. **A bar with heat generation** of constant rate H (> 0) is modeled by $u_t = c^2 u_{xx} + H$. Solve this problem if $L = \pi$ and the ends of the bar are kept at 0°C . *Hint:* Set $u = v - Hx(x - \pi)/(2c^2)$.

17. **Heat flux.** The heat flux of a solution $u(x, t)$ across $x = 0$ is defined by $\phi(t) = -Ku_x(0, t)$. Find $\phi(t)$ for the solution (9). Explain the name. Is it physically understandable that ϕ goes to 0 as $t \rightarrow \infty$?

18-25 TWO-DIMENSIONAL PROBLEMS

18. **Laplace equation.** Find the potential in the rectangle $0 \leq x \leq 20$, $0 \leq y \leq 40$ whose upper side is kept at potential 110 V and whose other sides are grounded.

19. Find the potential in the square $0 \leq x \leq 2$, $0 \leq y \leq 2$ if the upper side is kept at the potential $1000 \sin \frac{1}{2}\pi x$ and the other sides are grounded.

20. **CAS PROJECT. Isotherms.** Find the steady-state solutions (temperatures) in the square plate in Fig. 297 with $a = 2$ satisfying the following boundary conditions. Graph isotherms.

(a) $u = 80 \sin \pi x$ on the upper side, 0 on the others.

(b) $u = 0$ on the vertical sides, assuming that the other sides are perfectly insulated.

(c) Boundary conditions of your choice (such that the solution is not identically zero).

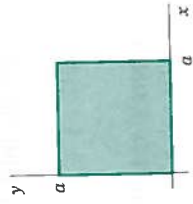


Fig. 297. Square plate

21. **Heat flow in a plate.** The faces of the thin square plate in Fig. 297 with side $a = 24$ are perfectly insulated. The upper side is kept at 25°C and the other sides are kept at 0°C . Find the steady-state temperature $u(x, y)$ in the plate.

22. Find the steady-state temperature in the plate in Prob. 21 if the lower side is kept at $U_0^\circ\text{C}$, the upper side at $U_1^\circ\text{C}$, and the other sides are kept at 0°C . *Hint:* Split into two problems in which the boundary temperature is 0 on three sides for each problem.

23. **Mixed boundary value problem.** Find the steady-state temperature in the plate in Prob. 21 with the upper and lower sides perfectly insulated, the left side kept at 0°C , and the right side kept at $f(y)^\circ\text{C}$.

24. **Radiation.** Find steady-state temperatures in the rectangle in Fig. 296 with the upper and left sides perfectly insulated and the right side radiating into a medium at 0°C according to $u_x(a, y) + hu(a, y) = 0$, $h > 0$ constant. (You will get many solutions since no condition on the lower side is given.)

25. Find formulas similar to (17), (18) for the temperature in the rectangle R of the text when the lower side of R is kept at temperature $f(x)$ and the other sides are kept at 0°C .