Polynomial interpolation

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The Python codes for this note are given in `polynomialinterpolation.py`.

1 Introduction

In the first part of Mathematics 4D/N you learned about the Fourier-series. By taking the sum of the series from 1 to N, you can find an *approximation* to a periodic function. Similar, polynomials can be used to approximate functions over some bounded interval $x \in [a, b]$. Such polynomials can be used for different purposes. The function itself may be unknown, and only measured data are available. In this case, a polynomial may be used to find approximations to intermediate values of the function. Polynomials are easy to integrate, and can be used to find approximations of integrals of more complicated functions. This will be exploited later in the course. And there are plenty of other applications.

In this part of the course, we will only discuss *interpolation polynomials*.

Interpolation problem.

Given $n+1$ points $(x_i, y_i)_{i=0}^n$. Find a polynomial $p(x)$ of lowest possible degree satisfying the *interpolation condition*

$$p(x_i) = y_i, \qquad i = 0, \dots, n. \tag{1}$$

The solution $p(x)$ is called the *interpolation polynomial*, the x_i values are called *nodes*, and the points (x_i, y_i) *interpolation points*.

Example 1: Given the points

x_i	0	2/3	1
y_i	1	1/2	0

.

The corresponding interpolation polynomial is

$$p_2(x) = (-3x^2 - x + 4)/4$$

The y-values of this example are chosen such that $y_i = \cos(\pi x_i/2)$. So $p_2(x)$ can be considered as an approximation to $\cos(\pi x/2)$ on the interval $[0, 1]$.

Content of this note. In this part, we will discuss the following:

- Method: How to compute the polynomials?
- Existence and uniqueness results.
- Error analysis: If the polynomial is used to approximate a function, how good is the approximation?

- Improvements: If the nodes x_i can be chosen freely, how should we do it in order to reduce the error?

1.1 Notation etc.

Let us start with some useful notation and facts about polynomials.

- A polynomial of degree n is given by

$$p_n(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x_1 + c_0, \qquad c_i \in \mathbb{R}, \quad i = 0, 1, \ldots, n. \tag{2}$$

- $\mathbb{P}_n$ is the set of all polynomials of degree n.
- $C^m[a, b]$ is the set of all continuous functions that have continuous first m derivatives.
- The value r is a root or a zero of a polynomial p if $p(r) = 0$.
- A nonzero polynomial of degree n can never have more than n real roots (there may be less).
- A polynomial of degree n with n real roots $r_1, r_2, \ldots, r_n$ can be written as

$$p_n(x) = c(x - r_1)(x - r_2) \cdots (x - r_n) = c \prod_{i=1}^{n} (x - r_i).$$

2 Methods

In this section, we present three techniques for finding the interpolation polynomial for a given set of data.

2.1 The direct approach

For a polynomial of degree n the interpolation condition (1) is a linear systems of $n + 1$ equations in $n + 1$ unknowns:

$$\sum_{i=0}^{n} x_j^i c_i = y_j, \qquad j = 0, \ldots, n.$$

If we are basically interested in the polynomials themself, given by the coefficients c_i, $i = 0, 1, \ldots, n$, this is a perfectly fine solution. It is for instance the strategy implemented in MATLAB's interpolation routines. However, in this course, polynomial interpolation will be used as a basic tool to construct other algorithms, in particular for integration. In that case, this is not the most convenient option, so we concentrate on a different strategy, which essentially makes it possible to just write up the polynomials.

2.2 Lagrange interpolation

Given $n + 1$ points $(x_i, y_i)_{i=0}^n$ with distinct x_i values. The *cardinal functions* are defined by:

$$\ell_i(x) = \prod_{j=0, j \neq i}^{n} \frac{x - x_j}{x_i - x_j} = \frac{x - x_0}{x_i - x_0} \cdots \frac{x - x_{i-1}}{x_i - x_{i-1}} \cdot \frac{x - x_{i+1}}{x_i - x_{i+1}} \cdots \frac{x - x_n}{x_i - x_n}, \qquad i = 0, \ldots, n.$$

The cardinal functions have the following properties:

- $\ell_i \in \mathbb{P}_n$, $i = 0, 1, \ldots, n$.
- $\ell_i(x_j) = \delta_{ij} = \begin{cases} 1, & \text{when } i = j \\ 0, & \text{when } i \neq j \end{cases}.$
- They are constructed solely from the nodes x_i's.
- They are linearly independent, and thus form a basis for $\mathbb{P}_n$.

The interpolation polynomial is now given by

$$p_n(x) = \sum_{i=0}^{n} y_i \ell_i(x)$$

since

$$p_n(x_j) = \sum_{i=0}^{n} y_i \ell_i(x_j) = y_j, \qquad j = 0, \ldots, n.$$

Example 2: Given the points:

$$\begin{array}{c|ccc} x_i & 0 & 1 & 3 \\ \hline y_i & 3 & 8 & 6 \end{array}.$$

The corresponding cardinal functions are given by:

$$\begin{aligned} \ell_0(x) &= \frac{(x-1)(x-3)}{(0-1)(0-3)} = \frac{1}{3}x^2 - \frac{4}{3}x + 1 \\ \ell_1(x) &= \frac{(x-0)(x-3)}{(1-0)(1-3)} = -\frac{1}{2}x^2 + \frac{3}{2}x \\ \ell_2(x) &= \frac{(x-0)(x-1)}{(3-0)(3-1)} = \frac{1}{6}x^2 - \frac{1}{6}x \end{aligned}$$

and the interpolation polynomial is given by (check it yourself):

$$p_2(x) = 3\ell_0(x) + 8\ell_1(x) + 6\ell_2(x) = -2x^2 + 7x + 3.$$

2.3 Implementation

The method above is implemented as two functions:

- `cardinal(xdata, x)`: Create a list of cardinal functions $\ell_i(x)$ evaluated in x.
- `lagrange(ydata, l)`: Create the interpolation polynomial $p_n(x)$.

Here, `xdata` and `ydata` are arrays with the interpolation points, and `x` is an array of values in which the polynomials are evaluated.

You are not required to understand the implementation of these functions, but you should understand how to use them.

Example 3: Test the functions on the interpolation points of Example 2.

See `example3()` in `polynomialinterpolation.py`.

Numerical exercises:

1. Plot the cardinal functions for the nodes of Example 1.
2. Plot the interpolation polynomials for some points of your own choice.

2.4 Newton interpolation

This is an alternative approach to find the interpolation polynomial.

Let $x_0, x_1, \ldots, x_n$ be $n+1$ distinct real numbers. The so-called Newton form of a polynomial of degree n is an expansion of the form

$$p(x) = \sum_{i=0}^{n-1} c_{n-i} \prod_{j=0}^{n-1-i} (x - x_j) + c_0,$$

or more explicitly

$$p(x) = c_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}) + c_{n-1}(x - x_0)(x - x_1)\cdots(x - x_{n-2}) + \cdots + c_1(x - x_0) + c_0.$$

In the light of this form of writing a polynomial, the polynomial interpolation problem leads to the following observations. Let us start with a single node x_0, then $f(x_0) = p(x_0) = c_0$. Going one step further and consider two nodes x_0, x_1. Then we see that $f(x_0) = p(x_0) = c_0$ and $f(x_1) = p(x_1) = c_0 + c_1(x_1 - x_0)$. The latter implies that the coefficient

$$c_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

Given three nodes x_0, x_1, x_2 yields the coefficients c_0, c_1 as defined above, and from

$$f(x_2) = p(x_2) = c_0 + c_1(x_2 - x_0) + c_2(x_2 - x_0)(x_2 - x_1)$$

we deduce the coefficient

$$c_2 = \frac{f(x_2) - f(x_0) - \frac{f(x_1)-f(x_0)}{x_1-x_0}(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)}.$$

Playing with this quotient gives the much more structured expression

$$c_2 = \frac{\frac{f(x_2)-f(x_1)}{x_2-x_1} - \frac{f(x_1)-f(x_0)}{x_1-x_0}}{(x_2 - x_0)}.$$

This procedure can be continued and yields a so-called triangular systems that permits to define the remaining coefficients $c_3, \ldots, c_n$. One sees quickly that the coefficient c_k only depends on the interpolation points $(x_0, y_0), \ldots, (x_k, y_k)$, where $y_i := f(x_i)$, $i = 0, \ldots, n$.

We introduce the folllwing so-called finite difference notation for a function f. The 0th order finite difference is defined to be $f[x_0] := f(x_0)$. The 1st order finite difference is

$$f[x_0, x_1] := \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

The second order finite difference is defined by

$$f[x_0, x_1, x_2] := \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}.$$

In general, the nth order finite difference of the function f is defined to be

$$f[x_0, \ldots, x_n] := \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}.$$

Newton's method to solve the polynomial interpolation problem can be summarized as follows. Given $n + 1$ interpolation points $(x_0, y_0), \ldots, (x_n, y_n)$, $y_i := f(x_i)$. If the order n interpolation polynomial is expressed in Newton's form

$$p_n(x) = c_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}) + c_{n-1}(x - x_0)(x - x_1)\cdots(x - x_{n-2}) + \cdots + c_1(x - x_0) + c_0,$$

then the coefficients

$$c_k = f[x_0, \ldots, x_k]$$

for $k = 0, \ldots, n$. In fact, a recursion is in place

$$p_n(x) = p_{n-1}(x) + f[x_0, \ldots, x_n](x - x_0)(x - x_1) \cdots (x - x_{n-1})$$

It is common to write the finite differences in a table, which for $n = 3$ will look like:

$$\begin{array}{c|cccc}
x_0 & f[x_0] & & & \\
 & & f[x_0, x_1] & & \\
x_1 & f[x_1] & & f[x_0, x_1, x_2] & \\
 & & f[x_1, x_2] & & f[x_0, x_1, x_2, x_3] \\
x_2 & f[x_2] & & f[x_1, x_2, x_3] & \\
 & & f[x_2, x_3] & & \\
x_3 & f[x_3] & & &
\end{array}$$

Example 1 again: Given the points in Example 1. The corresponding table of divided differences becomes:

$$\begin{array}{c|ccc}
0 & 1 & & \\
 & & -3/4 & \\
2/3 & 1/2 & & -3/4 \\
 & & -3/2 & \\
1 & 0 & &
\end{array}$$

and the interpolation polynomial becomes

$$p_2(x) = 1 - \frac{3}{4}(x - 0) - \frac{3}{4}(x - 0)(x - \frac{2}{3}) = 1 - \frac{1}{4}x - \frac{3}{4}x^2.$$

2.5 Implementation

The method above is implemented as two functions:

- `divdiff(xdata, ydata)`: Create the table of divided differences
- `newtonInterpolation(F, xdata, x)`: Evaluate the interpolation polynomial.

Here, `xdata` and `ydata` are arrays with the interpolation points, and `x` is an array of values in which the polynomial is evaluated. Run the code on the example above: See `example_divided_differences()` in `polynomialinterpolation.py`.

3 Theory

In this section we cover two theoretical aspects, and give the answer to one natural question:

- Do interpolation polynomials always exist, and when they exist, are they unique?
- If the polynomial is used to approximate a function, can we find an expression for the error?
- How can the error be made as small as possible?

3.1 Existence and uniqueness of interpolation polynomials.

We have already proved the existence of such polynomials, simply by constructing them. But are they unique?

Suppose there exist two different interpolation polynomials p_n and q_n of degree n interpolating the same $n + 1$ points. The polynomial $r(x) = p_n(x) - q_n(x)$ is of degree n with zeros in all the nodes x_i, that is a total of $n + 1$ zeros. But then $r \equiv 0$, and the two polynomials p_n and q_n are identical. We have then proved the following result:

Theorem: Existence and uniqueness.

Given $n+1$ points $(x_i, y_i)_{i=0}^n$ with distinct x values. Then there is one and only one polynomial $p_n(x) \in \mathbb{P}_n$ satisfying the interpolation condition

$$p_n(x_i) = y_i, \qquad i = 0, \dots, n.$$

3.2 Error Analysis

Given some function $f \in C[a, b]$. Choose $n+1$ distinct nodes in $[a, b]$ and let $p_n(x) \in \mathbb{P}_n$ satisfy the interpolation condition

$$p_n(x_i) = f(x_i), \qquad i = 0, \dots, n.$$

What can be said about the error $e(x) = f(x) - p_n(x)$?

Let us start with an numerical experiment, to have a certain feeling of what to expect.

Example 4: Let $f(x) = \sin(x)$, $x \in [0, 2\pi]$. Choose $n+1$ equidistributed nodes, that is $x_i = ih$, $i = 0, \dots, n$, and $h = 2\pi/n$. Calculate the interpolation polynomial by use of the functions `cardinal` and `lagrange`. Plot the error $e_n(x) = f(x) - p_n(x)$ for different values of n. Choose $n = 4, 8, 16$ and 32. Notice how the error is distributed over the interval, and find the maximum error $\max_{x\in[a,b]} |e_n(x)|$ for each n.

See `example4()` in `polynomialinterpolation.py`.

Numerical exercise:

- Repeat the experiment with Runge's function

$$f(x) = \frac{1}{1+x^2}, \qquad x \in [-5, 5].$$

Let us now see if we can find an expression for the error $e(x) = f(x) - p_n(x)$. Assume that the nodes $x_i \in [a, b]$, $i = 0, \dots, n$ are distinct, and that f is sufficiently differentiable, that is $f \in C^r[a, b]$. How differentiable will be clear as we go.

Define the function

$$\omega(x) = \prod_{i=0}^{n} (x - x_i) = x^{n+1} + \cdots .$$

Clearly, the error in the nodes, $e(x_i) = 0$. Choose an *arbitrary* $x \in [a, b]$, $x \in [a, b]$, where $x \neq x_i$, $i = 0, 1, \dots, n$. For this fixed x, define a function in t as:

$$\varphi(t) = e(t)\omega(x) - e(x)\omega(t).$$

where $e(t) = f(t) - p_n(t)$. Notice that $\varphi(t)$ is as differentiable with respect to t as $f(t)$. The function $\varphi(t)$ has $n+2$ distinct zeros (the nodes and the fixed x). As a consequence of Rolle's theorem, the derivative $\varphi'(t)$ has at least $n+1$ distinct zeros, one between each of the zeros of $\varphi(t)$. So $\varphi''(t)$ has n distinct zeros, etc. By repeating this argument, we can see that $\varphi^{n+1}(t)$ has at least one zero in $[a, b]$, let us call this $\xi(x)$, as it do depend on the fixed x. Since $\omega^{(n+1)}(t) = (n+1)!$ and $e^{(n+1)}(t) = f^{(n+1)}(t)$ we can conclude that

$$\varphi^{(n+1)}(\xi) = 0 = f^{(n+1)}(\xi)\omega(x) - e(x)(n+1)!,$$

The argument above is valid if f is $n+1$ continuous differentiable on $[a, b]$. Altogether we have proved the following:

Theorem: Interpolation error.

Given $f \in C^{(n+1)}[a,b]$. Let $p_n \in \mathbb{P}_n$ interpolate f in $n+1$ distinct nodes $x_i \in [a,b]$. For each $x \in [a,b]$ there is at least one $\xi(x) \in (a,b)$ such that

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^{n} (x - x_i).$$

The interpolation error consists of three elements: The derivative of the function f, the number of interpolation points $n+1$ and the distribution of the nodes x_i. We cannot do much with the first of these, but we can choose the two others. Let us first look at the most obvious choice of nodes.

Equidistributed nodes. The nodes are *equidistributed* over the interval $[a,b]$ if $x_i = a+ih$, $h = (b-a)/n$. In this case it can be proved that:

$$|\omega(x)| \leq \frac{h^{n+1}}{4} n!$$

such that

$$|e(x)| \leq \frac{h^{n+1}}{4(n+1)} M, \qquad M = \max_{x \in [a,b]} |f^{(n+1)}(x)|.$$

for all $x \in [a,b]$.

Let us now see how good this error bound is by an example.

Example 5: Let again $f(x) = \sin(x)$ and $p_n(x)$ the polynomial interpolating $f(x)$ in $n+1$ equidistributed points. Find an upper bound for the error for different values of n.

Clearly, $\max_{x \in [0,2\pi]} |f^{(n+1)}(x)| = M = 1$ for all n, so

$$|e_n(x)| = |f(x) - p_n(x)| \leq \frac{1}{4(n+1)} \left(\frac{2\pi}{n} \right)^{n+1}, \qquad x \in [a,b].$$

Use the code in Example 4 to verify the result. How close is the bound to the real error?

3.3 Optimal choice of interpolation points

So how can the error be reduced? For a given n there is only one choice: to distribute the nodes in order to make $|\omega(x)| = \prod_{j=0}^{n} |x - x_i|$ as small as possible. We will first do this on a standard interval $[-1,1]$, and then transfer the results to some arbitrary interval $[a,b]$.

Let us start taking a look at $\omega(x)$ for equidistributed nodes on the interval $[-1,1]$, for different values of n:

Run the function `plot_omega()`.

Run the code for different values of n. Notice the following:

- $\max_{x \in [-1,1]} |\omega(x)|$ becomes smaller with increasing n.
- $|\omega(x)|$ has its maximum values near the boundaries of $[-1,1]$.

A a consequence of the latter, it seems reasonable to move the nodes towards the boundaries. It can be proved that the optimal choice of nodes are the *Chebyshev-nodes*, given by

$$\tilde{x}_i = \cos\left(\frac{(2i+1)\pi}{2(n+1)}\right), \qquad i = 0, \dots, n$$

Let $\omega_{Cheb}(x) = \prod_{j=1}^{n}(x - \tilde{x}_i)$. It is then possible to prove that

$$\frac{1}{2^n} = \max_{x\in[-1,1]} |\omega_{Cheb}(x)| \leq \max_{x\in[-1,1]} |q(x)|$$

for all polynomials $q \in \mathbb{P}_n$ such that $q(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$.

The distribution of nodes can be transferred to an interval $[a, b]$ by the linear transformation

$$x = \frac{b-a}{2}\tilde{x} + \frac{b+a}{2}$$

where $x \in [a, b]$ and $\tilde{x} \in [-1, 1]$. By doing so we get

$$\omega(x) = \prod_{j=0}^{n}(x - x_i) = \left(\frac{b-a}{2}\right)^{n+1} \prod_{j=0}^{n}(\tilde{x} - \tilde{x}_i) = \left(\frac{b-a}{2}\right)^{n+1} \omega_{Cheb}(\tilde{x}).$$

From the theorem on interpolation errors we can conclude:

Theorem (interpolation error for Chebyshev interpolation).

Given $f \in C^{(n+1)}[a, b]$, and let $M_{n+1} = \max_{x\in[a,b]} |f^{(n+1)}(x)|$. Let $p_n \in \mathbb{P}_n$ interpolate f i $n + 1$ Chebyshev-nodes $x_i \in [a, b]$. Then

$$\max_{x\in[a,b]} |f(x) - p_n(x)| \leq \frac{(b-a)^{n+1}}{2^{2n+1}(n+1)!} M_{n+1}.$$

The Chebyshev nodes over an interval $[a, b]$ are evaluated in the following function: See `chebyshev_nodes(a, b, n)` in `polynomialinterpolation.py`.

Numerical exercises:

1. Plot $\omega_{Cheb}(x)$ for $3, 5, 9, 17$ interpolation points.
2. Repeat Example 3 using Chebyshev interpolation on the functions below. Compare with the results you got from equidistributed nodes.

$$f(x) = \sin(x), \qquad x \in [0, 2\pi]$$

$$f(x) = \frac{1}{1+x^2}, \qquad x \in [-5, 5].$$

For information: Chebfun is software package which makes it possible to manipulate functions and to solve equations with accuracy close to machine accuracy. The algorithms are based on polynomial interpolation in Chebyshev nodes.