Norwegian University of Science and Technology
Department of Mathematics

TMA4145 Linear Methods Autumn 2012

Exercise set 10

1 Let $A \in M_{n\times n}(\mathbb{R})$. Use Picard iteration for $\dot{x} = Ax$, $x(0) = x_0$, to find the solution of this initial-value problem.

2 (*Friedberg 5.1:8, 9*) Let $A \in M_{n\times n}$ be a real or complex matrix.

a) Show that a A is invertible if and only if zero is not an eigenvalue of A.

b) Assume that A is invertible. Prove that λ is an eigenvalue of A if and only if λ^{-1} is an eigenvalue of A^{-1}.

c) Assume that A is upper triangular. Prove that its eigenvalues are its diagonal entries.

3 (*Friedberg 5.2:2, 7.2:4*) Determine the eigenvalues and generalized eigenspaces of the following matrices. Which of these matrices are diagonalizable? For each matrix, find its Jordan normal form and present the ordered basis which gives this form.

a)

$$\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

b)

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

c)

$$\begin{bmatrix} 0 & 1 & -1 \\ -4 & 4 & -2 \\ -2 & 1 & 1 \end{bmatrix}$$

4 (*Friedberg 5.2:14*) Show, by constructing them, that the general solutions of the following systems of differential equations are as suggested.

a)

$$x(t) = c_1 e^{3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{for} \quad \begin{cases} \dot{x}_1 = 8x_1 + 10x_2, \\ \dot{x}_2 = -5x_1 - 7x_2. \end{cases}$$

b)

$$x(t) = e^t \left(c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) + c_3 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \text{for} \qquad \begin{cases} \dot{x}_1 = x_1 + x_3, \\ \dot{x}_2 = x_2 + x_3, \\ \dot{x}_3 = 2x_3. \end{cases}$$

$\boxed{5}$ (*Jacobi iteration*) Given $A \in M_{n\times n}(\mathbb{R})$ with non-zero diagonal elements (a_{ii}), let the diagonal matrix $D \in M_{n\times n}(\mathbb{R})$ be defined by

$$\begin{cases} d_{ij} = a_{ij}, & i = j, \\ d_{ij} = 0, & i \neq j, \end{cases} \qquad \text{for} \qquad i, j = 1, \ldots, n.$$

Then

$$Ax = c \quad \Longleftrightarrow \quad Dx = (D - A)x + c \quad \Longleftrightarrow \quad x = D^{-1}(D - A)x + D^{-1}c.$$

a) Use Problem 5 from Problem set 9 to find a condition on A that makes this system solvable with fixed-point arguments.

Hint: In case of difficulties, consult Kreyszig.

b) Given that you relied on Problem 5 from Problem set 9 to solve (a), show that the set of matrices A satisfying this condition is an open set in $M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{nn}$.

$\boxed{6}$ (*Challenge*) For $f\colon \mathbb{R} \to \mathbb{R}$, let T^t denote the shift operator $(T^t f)(x) := f(x + t)$.

a) Consider the case when $f(x) = x^k$ is a monomial, to show that the representation $T^t = e^{t\frac{d}{dx}}$ makes sense (expand the exponential function in an infinite series).

Hint: the Binomial Theorem.

b) If the *Fourier transform* $(\mathcal{F}f)(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x)\,dx$ of f exists, and assuming that this integral is absolutely convergent, determine $\mathcal{F}T^t f$.

Hint: compare your result with (a).

c) Given that the Fourier transform $\mathcal{F}$ is a bounded linear bijection $L_2(\mathbb{R}, \mathbb{C}) \to L_2(\mathbb{R}, \mathbb{C})$ with bounded inverse, and that the result from (b) can be extended to $f \in L_2(\mathbb{R}, \mathbb{C})$, show that $T^t\colon L_2(\mathbb{R}, \mathbb{C}) \to L_2(\mathbb{R}, \mathbb{C})$ has no eigenvalues outside the unit circle.

Additional suggestions: Problem 4 from the final 2004 is perfect repetition of this week's material. See also Problem 5 from the final 2000.