Norwegian University of Science and Technology
Department of Mathematics

TMA4145 Linear Methods
Autumn 2012

Exercise set 11

1 (*Young: Exercises 1.3, 1.10, Problem 1.8*)

a) Show that

$$\langle f, g\rangle = \int_0^1 f(t)\overline{g(t)}\,dt$$

defines an inner product on $C([0,1],\mathbb{C})$, the space of complex-valued continuous functions on the interval $[0,1]$.

b) Prove that, for any $f \in C([0,1],\mathbb{C})$,

$$\left|\int_0^1 f(t)\sin(\pi t)\,dt\right| \leq \frac{1}{\sqrt{2}}\left(\int_0^1 |f(t)|^2\,dt\right)^{1/2},$$

and classify the functions f for which equality holds.

c) Prove that, for any $f \in C^1([-\pi,\pi],\mathbb{C})$,

$$\left|\int_{-\pi}^{\pi} f(t)\cos(t) - f'(t)\sin(t)\,dt\right| \leq \sqrt{2\pi}\left(\int_{-\pi}^{\pi} |f(t)|^2 + |f'(t)|^2\,dt\right)^{1/2}.$$

Hint: you need to generalize the inner product from (a) here.

2 (*Parallelogram law and Polarization identity*) For a normed space $(X, \|\cdot\|)$, the parallelogram law is the identity

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

a) Show that if X is a real space, and the parallelogram law holds, then

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$$

defines an inner product on X satisfying $\langle x, x\rangle = \|x\|^2$.[1]

Hint: To prove $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$ first prove that $\langle 2x, y\rangle = 2\langle x, y\rangle$, then generalize this to $\langle mx, y\rangle = m\langle x, y\rangle$ and $\langle \frac{m}{n}x, y\rangle = \frac{m}{n}\langle x, y\rangle$ for positive integers $m, n \in \mathbb{N}$. Use a continuity/limiting argument to pass to $\lambda > 0$. Finally, deduce the desired equality by using that $\lambda x = |\lambda|(-x)$ for negative λ.

b) Give an example of a normed space $(X, \|\cdot\|)$ and vectors $x, y \in X$ such that the parallelogram law does not hold for x, y.

[1] Similarly, if X is a complex space, and the parallelogram law holds, one can show that $\langle x, y\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k\|x + i^k y\|^2$ defines an inner product on X satisfying $\langle x, x\rangle = \|x\|^2$.

3 (*Problem 4, 2004*) The matrix

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

has characteristic polynomial $p_A(\lambda) = (3 - \lambda)^3$.

a) Find a Jordan form J for A.

b) Find a matrix S such that $S^{-1}AS = J$.

c) Find the general solution $u\colon \mathbb{R} \to \mathbb{R}^3$ of the differential equation $\dot{u} = Au$.

4 (*Young: Problem 3.10*)

a) Let M be a non-empty closed and convex set in a Hilbert space. Show that M contains a unique vector $x_{\min}$ of smallest norm, and that

$$\operatorname{Re} \langle x_{\min}, x_{\min} - x \rangle \le 0 \qquad \text{for all} \quad x \in M.$$

Hint: To obtain the inequality, use that $x_{min} - t(x_{min} - x) \in M$ by convexity for arbitrarily small $t > 0$.

b) Try to find a nice example in $\mathbb{R}^2$ which exemplifies (a). Illustrate with a picture.

5 (*Young: Problems 3.7, 3.8*)

a) Show that the unit ball in a normed space is convex; then show that all open balls in a normed space are convex.

b) Show that, in a normed space, the closure of any convex set is convex. Deduce that all balls (open or closed) in a normed space are convex.

c) Show that in l_∞ the closed linear subspace c_0 (which is a convex set) has no unique element that minimizes the distance to the element $(1, 1, 1, \ldots) \in l_\infty$ consisting of only ones. You can do this for l_∞ real or complex, as you like.

6 (*Challenge*) Show that $C([0, 1], \mathbb{R})$ with the L_2-inner product is not a Hilbert space.