Norwegian University of Science and Technology
Department of Mathematics

TMA4145 Linear Methods Autumn 2012

Exercise set 12

1 (*Young, Problem 4.1*) Find a vector in $\mathbb{C}^3$ orthogonal to $(1,1,1)$ and $(1,\omega,\omega^2)$, where $\omega = e^{2\pi i/3}$. (The inner product in $\mathbb{C}^3$ is $\langle z, w\rangle = \sum_{j=1}^3 z_j \overline{w_j}$.)

2 (*Problem 3, 2005*) Let $E = C([0,1],\mathbb{C})$ be the linear space of complex-valued continuous functions on $[0,1]$. Define

$$\|x\|_\infty = \max_{t\in[0,1]} |x(t)| \quad \text{and} \quad \langle x, y\rangle = \int_0^1 x(t)\overline{y(t)}\,dt,$$

where $x, y \in E$.

a) Show that $\|\cdot\|_\infty$ is a norm on E, and that $\langle\cdot,\cdot\rangle$ is an inner product on E.

b) Show that $M := \{x \in E : x(0) = 0\}$ is a closed linear subspace of E with respect to the metric d induced by $\|\cdot\|_\infty$, but that M is not closed with respect to the metric $\tilde{d}$ induced by $\langle\cdot,\cdot\rangle$.

c) Show that the linear functional $\phi\colon E \to \mathbb{C}$, defined by $\phi(x) = x(0)$, $x \in E$, is continuous with respect to the metric d, but discontinuous with respect to the metric $\tilde{d}$ on E.

d) Find $a, b \in \mathbb{C}$ such that

$$\int_0^1 |t^4 - a - bt|^2\,dt$$

is minimal.

3 (*Problem 4, 2005*) The set $C([-\frac{1}{2},\frac{1}{2}],\mathbb{R})$ of real-valued continuous functions on $[-\frac{1}{2},\frac{1}{2}]$ with the metric

$$d(x,\tilde{x}) = \max_{t\in[-\frac{1}{2},\frac{1}{2}]} |x(t) - \tilde{x}(t)|$$

is a complete metric space.[1] The function $f(t,u) = u^3 - 2t^2$ satisfies the Lipschitz condition

$$|f(t,u) - f(t,v)| \le \frac{3}{4}|u - v| \quad \text{in} \quad R = [-\tfrac{1}{2},\tfrac{1}{2}] \times [-\tfrac{1}{2},\tfrac{1}{2}].$$

a) Let $X = \{x \in C([-\frac{1}{2},\frac{1}{2}],\mathbb{R}) \colon d(x,0) \le \frac{1}{2}\}$, and define T by

$$(Tx)(t) = \int_0^t f(\tau, x(\tau))\,d\tau.$$

Show that T maps X into X.

[1] It coincides with $BC([-\frac{1}{2},\frac{1}{2}],\mathbb{R})$.

b) Show that $T\colon X \to X$ is a contraction with contraction constant $\lambda = \frac{3}{8}$.

c) Let $x_0 = 0$, $x_1 = Tx_0$, and $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. Explain why the sequence $\{x_n\}_{n=1}^{\infty}$ converges (with respect to the metric d) to a function $\tilde{x} \in C([0,1],\mathbb{R})$, and give an argument for why $\tilde{x} = \tilde{x}(t)$ satisfies the differential equation

$$x' = x^3 - 2t^2, \qquad t \in \left(-\tfrac{1}{2}, \tfrac{1}{2}\right)$$

with the initial condition $x(0) = 0$.

4 (*Young, Problem 4.6: The Gram–Schmidt process*) Let $x_1, x_2, \ldots$ be a sequence of linearly independent vectors in an inner-product space. Define vectors $e_1, e_2, \ldots$ inductively as follows:

$$e_1 = x_1/\|x_1\|,$$
$$f_n = x_n - \sum_{j=1}^{n-1} \langle x_n, e_j\rangle e_j, \qquad e_n = f_n/\|f_n\|, \qquad n \geq 2.$$

Show that $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal sequence having the same closed linear span as $\{x_n\}_{n\in\mathbb{N}}$.

5 (*Problem 5, 2008*) Let $C([0,1],\mathbb{C})$ denote the space of complex-valued continuous functions on $[0,1]$, and let

$$\langle f, g\rangle = \int_0^1 f(t)\overline{g(t)}\, dt$$

be the standard inner product on $C([0,1],\mathbb{C})$.

a) Let $T\colon C([0,1],\mathbb{C}) \to C([0,1],\mathbb{C})$ be the linear transformation which sends $f \in C([0,1],\mathbb{C})$ to the function Tf given by $(Tf)(t) = tf(t)$. Show that T is *self-adjoint*, i.e., that

$$\langle Tf, g\rangle = \langle f, Tg\rangle$$

for all $f, g \in C([0,1],\mathbb{C})$.

b) For $f, g \in C([0,1],\mathbb{C})$ define $\langle f, g\rangle_T$ by

$$\langle f, g\rangle_T = \langle Tf, g\rangle.$$

Prove that $\langle\cdot,\cdot\rangle_T$ is an inner product on $C([0,1],\mathbb{C})$.

c) Let $V \subset C([0,1],\mathbb{C})$ be the subspace $\{a + bt + ce^t \colon a, b, c \in \mathbb{C}\}$. Find a and b such that

$$\int_0^1 t|e^t - a - bt|^2\, dt$$

is minimal.