Norwegian University of Science
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Department of Mathematics

TMA4145 Linear Methods
Autumn 2012

Exercise set 13

$\boxed{1}$ (*Problem 4, 2000*)

a) Find the QR-factorization of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}.$$

b) Which point in $\operatorname{span}\{(1,1,1),(1,2,3)\} \subset \mathbb{C}^3$ is closest to $(\mathrm{i},0,1)$?

$\boxed{2}$ (*Problem 6, 2001*)

a) Let $T \in B(H)$ be a bounded linear transformation $H \to H$. Show that

$$\ker(T) = \ker(T^*T).$$

b) Let $A \in M_{m\times n}(\mathbb{C})$. Show that

$$\ker(A) = \ker(A^*A).$$

$\boxed{3}$ (*Problem 3, 2002*)

a) Let $A \in M_{n\times n}(\mathbb{R})$. Prove that

$$\langle x, y\rangle_A = y^t A x$$

defines an inner product on $\mathbb{R}^n$ if and only if A is symmetric and positive definite.

b) Let $v_1(t) = e^t$, $v_2(t) = e^{-t}$, and let V be the function space $\{av_1 + bv_2 \colon a, b \in \mathbb{R}\}$ with inner product

$$\langle x, y\rangle = \int_0^1 x(t)y(t)\,dt.$$

Furthermore, let $T\colon V \to \mathbb{R}^2$ be the linear transformation

$$T(av_1 + bv_2) = \begin{bmatrix} a \\ b \end{bmatrix}.$$

Find $A \in M_{2\times 2}(\mathbb{R})$ such that T is unitary on $(\mathbb{R}^2, \langle\cdot,\cdot\rangle_A)$.

$\boxed{4}$ (*Problem 4, 2003*) Let $A \in M_{n\times n}(\mathbb{C})$. Show that if A can be diagonalized by a unitary matrix, then $|Ax| = |A^*x|$ for all $x \in \mathbb{C}^n$.

$\boxed{5}$ (*Problem 3, 2011*) Let $T \in L(\mathbb{R}^4)$, and suppose that there are $a, b \in \mathbb{R}$ such that T is realized by the matrix

$$\begin{bmatrix} 1 & 1 & 3 & a \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & b \end{bmatrix}.$$

a) Find the eigenvalues and eigenvectors of T in terms of a and b.

b) For which values of a and b does $\mathbb{R}^4$ have a basis of eigenvectors of T?

c) Are there choices for a and b so that $\mathbb{R}^4$ has an orthonormal basis of eigenvectors of T?

$\boxed{6}$ (*Problem 1, 2002*) Let

$$A = \frac{\sqrt{2}}{10} \begin{bmatrix} 3 & 0 & -4 \\ 0 & 10 & 0 \\ -3 & 0 & 4 \\ 0 & 10 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \frac{1}{25} \begin{bmatrix} 9 & 0 & -12 \\ 0 & 100 & 0 \\ -12 & 0 & 16 \end{bmatrix},$$

so that $A^t A = B$.

a) Give an orthogonal diagonalization QDQ^t of B.

b) Find $\exp(tB)$ and solve the initial-value problem

$$\dot{x} = Bx, \qquad x(0) = \begin{bmatrix} 0 \\ 25 \\ 50 \end{bmatrix}.$$

c) Find the singular value decomposition and the pseudoinverse of A.