Norwegian University of Science and Technology
Department of Mathematics

TMA4145 Linear Methods
Autumn 2012

Exercise set 2

$\boxed{1}$ *(Kreyzig, exercise 1.1.2)* Does $d(x, y) = (x - y)^2$ define a metric on the real line? Why/why not?

$\boxed{2}$ *(Kreyzig, exercise 1.1.10)* Let X be the set of all ordered triples of zeros and ones.

a) What is the cardinality of X?

b) Let $d(x, y) =$ 'number of places where x and y have different entries'. Show that d is a metric on X.

The metric d is called the *Hamming distance* and has applications in coding theory, telecommunications and automata theory.

$\boxed{3}$ *(Friedberg exercise 1.2.12)* A function $f : \mathbb{R} \to \mathbb{R}$ is called *even* if $f(t) = f(-t)$ for all $t \in \mathbb{R}$.

Prove that the set of even functions $\mathbb{R} \to \mathbb{R}$ with component-wise addition,

$$(f + g)(t) \stackrel{def.}{=} f(t) + g(t),$$

and scalar multiplication,

$$(\lambda f)(t) \stackrel{def.}{=} \lambda(f(t)), \qquad \lambda \in \mathbb{R},$$

is a real vector space.
Hint: you need to find the zero vector, the additive inverse of an element f, and check the axioms of a real vector space. Most importantly, check that the operations of addition and scalar multiplication does not 'lead out of the space', i.e., that they are operations $X \times X \to X$ and $\mathbb{R} \times X \to X$, respectively.

$\boxed{4}$ *(Kreyzig, exercise 1.1.12)* Let (X, d) be metric space X.

a) Use the *generalized triangle inequality*[1]

$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{n-1}, x_n), \qquad x_j \in X,$$

to show that

$$|d(x, y) - d(z, w)| \leq d(x, z) + d(y, w), \qquad x, y, z, w \in X.$$

[1] The generalized triangle inequality follows from the triangle inequality by induction on n.

b) For $x, y, z, w \in X$, is it always true that

$$d(x,y) - d(z,w) \leq d(x,z) + d(y,w)?$$

$\boxed{5}$ **a)** Prove that

$$\|f\|_\infty = \sup_{t\in(0,1)} |f(t)|$$

defines a norm on the space of bounded and continuous functions $BC((0,1),\mathbb{R})$.

b) Which of the following functions are in $BC((0,1),\mathbb{R})$:

$$(i)\ t \mapsto \frac{1}{t}, \qquad (ii)\ t \mapsto \sin(t), \qquad (iii)\ t \mapsto \sin(1/t)?$$

$\boxed{6}$ *(Challenge)* Let $C^\infty_{2\pi-\mathrm{per}}(\mathbb{R},\mathbb{R})$ be the space of smooth (i.e., infinitely differentiable), 2π-periodic real functions, and consider the ordinary differential equations

$$f + f'' = 0, \tag{1}$$
$$f - f'' = 0. \tag{2}$$

a) Find the general solution of (1) and (2), respectively.

This shows that $1 + \left(\frac{d}{dx}\right)^2 : C^\infty_{2\pi-\mathrm{per}}(\mathbb{R},\mathbb{R}) \to C^\infty_{2\pi-\mathrm{per}}(\mathbb{R},\mathbb{R})$ is not injective.

b) Let

$$g(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$$

be the Fourier series of a function $g \in C^\infty_{2\pi-\mathrm{per}}(\mathbb{R},\mathbb{R})$. Can you find f such that $f - f'' = g$?

In finding f, you find the inverse of $1 - \left(\frac{d}{dx}\right)^2$ on $C^\infty_{2\pi-\mathrm{per}}(\mathbb{R},\mathbb{R})$.

c) Find a function $g \in C^\infty_{2\pi\text{-per}}(\mathbb{R},\mathbb{R})$ such that $f + f'' = g$ has no solution $f \in C^\infty_{2\pi\text{-per}}(\mathbb{R},\mathbb{R})$. *Hint: write g and f as Fourier series. Recall (a).*

This shows that $1 + \left(\frac{d}{dx}\right)^2 : C^\infty_{2\pi-\mathrm{per}}(\mathbb{R},\mathbb{R}) \to C^\infty_{2\pi-\mathrm{per}}(\mathbb{R},\mathbb{R})$ is not surjective.