TMA4145 Linear Methods
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Exercise set 4

1 *(Friedberg 1.2.18)* Consider the set $\mathbb{R}^2$ and define addition and scalar multiplication on it as follows.

$$(x_1, x_2) + (y_1, y_2) = (x_1 + 2y_1, x_2 + 3y_2) \qquad \lambda(x_1, x_2) = (\lambda x_1, \lambda x_2)$$

Prove or disprove that $\mathbb{R}^2$ under these operations is a vector space.

2 *(Kreyzig 1.5.3)* Show that the subspace $X \subset BC([a, b], \mathbb{R})$ defined by

$$X = \{f \in BC([a, b], \mathbb{R}) : f(a) = f(b)\}$$

is complete.

3 *(Kreyzig 2.2.4)* We consider the definition of a norm on a real vector space X.

a) Show that replacing the condition

$$\|x\| = 0 \Leftrightarrow x = 0$$

with

$$\|x\| = 0 \Rightarrow x = 0$$

does not alter the the concept of a norm (a norm under the "new axioms" will fulfill the "old axioms" as well).

b) Show that the axioms

$$\|\lambda x\| = |\lambda|\|x\| \qquad \lambda \in \mathbb{R}, x \in X$$

and

$$\|x + y\| \leq \|x\| + \|y\| \qquad x, y \in X$$

implies the non-negativity of a norm.

4 *(Kreyzig 2.2.13)* Let $X \neq \{0\}$ be a vector space. Show that the discrete metric on X cannot be obtained from any norm on X (There is no norm $\|\cdot\|$ on X such that $d(x, y) = \|x - y\|$ is the discrete metric).

Definition.

$$l_\infty(\mathbb{R}) = \{(x_i)_{i\in\mathbb{N}} : x_i \in \mathbb{R}, \sup_{i\in\mathbb{N}} |x_i| < \infty\}$$

The norm on $l_\infty(\mathbb{R})$ *is* $\|x\|_{l^\infty} = \sup_{i\in\mathbb{N}} |x_i|$

5 *(Kreyzig 1.5.8)* Let $M \subset l_\infty(\mathbb{R})$ be the subspace consisting of all sequences $x = (x_i)_{i\in\mathbb{N}}$ with finitely many nonzero terms.

Find a Cauchy sequence in M that does not converge in M.

NB: Each element in M is a sequence, and you are looking for a sequence of elements of M; thus you are looking for a sequence of sequences.

6 *(Challenge)*

a) Prove that $l_\infty(\mathbb{R})$ is complete.

b) Show that $l_\infty(\mathbb{R})$, the space of bounded sequences in $\mathbb{R}$, equipped with the infinity norm, is not separable.

Hint: assume you have a countable sequence (of sequences) that is dense in l_∞ and construct an element (i.e. a sequence) that has a distance greater than or equal to 1 to all elements of the sequence.