TMA4145 Linear Methods Autumn 2012

Exercise set 5

1 *(Friedberg 1.3.8)* Determine which of the following sets are subspaces of $\mathbb{R}^3$ under the operations of addition and scalar multiplication defined on $\mathbb{R}^3$. Justify your answers.

a) $\{(x, y, z) \in \mathbb{R}^3 : x = 3y \text{ and } z = -y\}$

b) $\{(x, y, z) \in \mathbb{R}^3 : x = z + 2\}$

c) $\{(x, y, z) \in \mathbb{R}^3 : 2x - 7y + z = 0\}$

d) $\{(x, y, z) \in \mathbb{R}^3 : x + 2y - 3z = 1\}$

e) $\{(x, y, z) \in \mathbb{R}^3 : 5x^2 - 3y^2 + 6z^2 = 0\}$

2 **a)** Show that the vectors (x, y, z) in a plane $P = \{(x, y, z) \in \mathbb{R}^3 \colon Ax + By + Cz = 0\}$, $A, B, C \in \mathbb{R}$, constitute a subspace of $\mathbb{R}^3$. (You only need to show that P is closed under addition and scalar multiplication; the vector space axioms follows from those of $\mathbb{R}^3$)

b) What are the possible values of $\dim P$, the dimension of P?

3 *(Strang 2.1.11)*Let $M_{2\times2}(\mathbb{R})$ be the set of real 2×2-matrices. $M_{2\times2}(\mathbb{R})$ is a real vector space under the following operations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix} \qquad \lambda \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{bmatrix}$$

Describe the smallest subspace of $M_{2\times2}(\mathbb{R})$ that contains

a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

4 *(Friedberg 1.6.1)* True or false?

a) The vector space $\{0\}$ has no basis.

b) Every vector space generated by a finite set has a basis.

c) Every vector space has a finite basis.

d) A vector space cannot have more that one basis.

e) If a vector space has a finite basis, then the number of vectors in any basis for the space is the same.

f) Suppose that V is a finitely generated vector space, that L is a linearly independent subset of V and that G is a subset generating V. Then L cannot contain more vectors than G.

g) If S generates the vector space V, then every vector in V can be written as a linear combination of vectors in S in only one way.

h) Every subspace of a finite-dimensional space is finite-dimensional.

i) If V is a vector space of dimension n, then V has exactly one subspace of dimension n and one subspace of dimension 0.

j) If V is a vector space of dimension n and S is a subset of V of cardinality n, then S is linearly independent if and only if S spans V.

5 *(Friedberg 1.6.3)* Which of the following sets are bases for $P_2(\mathbb{R})$, the real vector space of polynomials of degree at most 2?

a) $\{-1 - x + 2x^2, 2 + x - 2x^2, 1 - 2x + 4x^2\}$

b) $\{1 + 2x + x^2, 3 + x^2, x + x^2\}$

c) $\{1 - 2x - 2x^2, -2 + 3x - x^2, 1 - x + 6x^2\}$

d) $\{-1 + 2x + 4x^2, 3 - 4x - 10x^2, -2 - 5x - 6x^2\}$

e) $\{1 + 2x - x^2, 4 - 2x + x^2, -1 + 18x - 9x^2\}$

6 *(Friedberg 1.6.13)* The set of solutions to the system of linear equations

$$x_1 - 2x_2 + x_3 = 0$$
$$2x_1 - 3x_2 + x_3 = 0$$

is a subspace of $\mathbb{R}^3$. Find a basis for this subspace.

7 *Challenge* Show that the coordinates of the function f given by

$$f(\theta) := \sin^3(\theta) + \sin(\theta)\cos(\theta)$$

in the Schauder basis $\{1, \sin(\cdot), \cos(\cdot), \sin(2\cdot), \cos(2\cdot), \ldots\}$ for $L_2((-\pi, \pi), \mathbb{R})$ is

$$(0, \frac{3}{4}, 0, \frac{1}{2}, 0, -\frac{1}{4}, 0, 0, 0, \ldots).$$

(Hint: It is not necessary, but possible, to use Fourier theory to solve this problem.)